Math 123 Homework Assignment #2  
Due Friday, April 22

Part I:

1. Suppose that $A$ is a $C^*$-algebra.

   (a) Suppose that $e \in A$ satisfies $xe = x$ for all $x \in A$. Show that $e = e^*$ and that $\|e\| = 1$. Conclude that $e$ is a unit for $A$.

   (b) Show that for any $x \in A$, $\|x\| = \sup_{\|y\| \leq 1} \|xy\|$. (Do not assume that $A$ has an approximate identity.)

2. Suppose that $A$ is a Banach algebra with an involution $x \mapsto x^*$ that satisfies $\|x\|^2 \leq \|x^*x\|$. Then show that $A$ is a Banach *-algebra (i.e., $\|x^*\| = \|x\|$). In fact, show that $A$ is a $C^*$-algebra.

3. Let $I$ be a set and suppose that for each $i \in I$, $A_i$ is a $C^*$-algebra. Let $\bigoplus_{i \in I} A_i$ be the subset of the direct product $\prod_{i \in I} A_i$ consisting of those $a \in \prod_{i \in I} A_i$ such that $\|a\| := \sup_{i \in I} \|a_i\| < \infty$. Show that $(\bigoplus_{i \in I} A_i, \| \cdot \|)$ is a $C^*$-algebra with respect to the usual pointwise operations:

   $$(a + \lambda b)(i) := a(i) + \lambda b(i)$$
   $$(ab)(i) := a(i)b(i)$$
   $$a^*(i) := (a(i))^*.$$ 

   We call $\bigoplus_{i \in I} A_i$ the direct sum of the $\{A_i\}_{i \in I}$.

4. Let $A^1$ be the vector space direct sum $A \oplus \mathbb{C}$ with the *-algebra structure given by

   $$(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda \mu)$$
   $$(a, \lambda)^* := (a^*, \bar{\lambda}).$$

   Show that there is a norm on $A^1$ making it into a $C^*$-algebra such that the natural embedding of $A$ into $A^1$ is isometric. (Hint: If $1 \in A$, then show that $(a, \lambda) \mapsto (a + \lambda 1_A, \lambda)$ is a *-isomorphism of $A^1$ onto the $C^*$-algebra direct sum of $A$ and $\mathbb{C}$. If $1 \notin A$, then for each $a \in A$, let $L_a$ be the linear operator on $A$ defined by left-multiplication by $a$: $L_a(x) = ax$. Then show that the collection $B$ of operators on $A$ of the form $\lambda I + L_a$ is a $C^*$-algebra with respect to the operator norm, and that $a \mapsto L_a$ is an isometric *-isomorphism.)
5. In this question, ideal always means ‘closed two-sided ideal.’

(a) Suppose that $I$ and $J$ are ideals in a $C^*$-algebra $A$. Show that $IJ$ — defined to be the closed linear span of products from $I$ and $J$ — equals $I \cap J$.

(b) Suppose that $J$ is an ideal in a $C^*$-algebra $A$, and that $I$ is an ideal in $J$. Show that $I$ is an ideal in $A$.

6. Suppose that $a$ and $b$ are elements in a $C^*$-algebra $A$ and that $0 \leq a \leq b$. Show that $\|a\| \leq \|b\|$. What happens if we drop the assumption that $0 \leq a$? (Hint: use Lemma Z.)

Part II:

7. Suppose that $A$ is a unital $C^*$-algebra and that $f : \mathbb{R} \rightarrow \mathbb{C}$ is continuous. Show that the map $x \mapsto f(x)$ is a continuous map from $A_{\text{s.a.}} = \{ x \in A : x = x^* \}$ to $A$.

8. Prove Corollary AC: Show that every separable $C^*$-algebra contains a sequence which is an approximate identity. (Recall that we showed in the proof of Theorem AB that if $x \in A_{\text{s.a.}}$, and if $x \in \{ x_1, \ldots, x_n \} = \lambda$, then $\|x - xe\|^2 < 1/4n$.)

9. Suppose that $\pi : A \rightarrow B(\mathcal{H})$ is a representation. Prove that the following are equivalent.

(a) $\pi$ has no non-trivial closed invariant subspaces; that is, $\pi$ is irreducible.

(b) The commutant $\pi(A)' := \{ T \in B(\mathcal{H}) : T\pi(a) = \pi(a)T \text{ for all } a \in A \}$ consists solely of scalar multiples of the identity; that is $\pi(A)' = CI$.

(c) No non-trivial projection in $B(\mathcal{H})$ commutes with every operator in $\pi(A)$.

(d) Every vector in $\mathcal{H}$ is cyclic for $\pi$.

(Suggestions. Observe that $\pi(A)'$ is a $C^*$-algebra. If $A \in \pi(A)'_{\text{s.a.}}$ and $A \neq \alpha I$ for some $\alpha \in \mathbb{C}$, then use the Spectral Theorem to produce nonzero operators $B_1, B_2 \in \pi(A)'$ with $B_1B_2 = B_2B_1 = 0$. Observe that the closure of the range of $B_1$ is a non-trivial invariant subspace for $\pi$.)
Part III:

10. As in footnote 1 of problem #8 on the first assignment, use the maximum modulus theorem to view the disk algebra, $A(D)$, as a Banach subalgebra of $C(T)$.\(^1\) Let $f \in A(D)$ be the identity function: $f(z) = z$ for all $z \in T$. Show that $\sigma_{C(T)}(f) = T$, while $\sigma_{A(D)}(f) = \overline{D}$. This shows that, unlike the case of $C^*$-algebras where we have “spectral permanence,” we can have $\sigma_A(b)$ a proper subset of $\sigma_B(b)$ when $B$ is a unital subalgebra of $A$.

11. Suppose that $U$ is an bounded operator on a complex Hilbert space $H$. Show that the following are equivalent.

(a) $U$ is isometric on $\ker(U)^\perp$.
(b) $U^*U = U$.
(c) $UU^*$ is a projection\(^2\).
(d) $U^*U$ is a projection.

An operator in $B(H)$ satisfying (a), and hence (a)–(d), is called a partial isometry on $H$. The reason for this terminology ought to be clear from part (a).

Conclude that if $U$ is a partial isometry, then $UU^*$ is the projection on the (necessarily closed) range of $U$, that $U^*U$ is the projection on the $\ker(U)^\perp$, and that $U^*$ is also a partial isometry.

(Hint: Replacing $U$ by $U^*$, we see that (b)$\iff$(c) implies (b)$\iff$(c)$\iff$(d). Then use (b)–(d) to prove (a). To prove (c)$\iff$(b), consider $(UU^*U - U)(UU^*U - U)^*$.)

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\(^1\)Although it is not relevant to the problem, we can put an involution on $C(T)$, $f^*(z) = \overline{f(\overline{z})}$, making $A(D)$ a Banach $*$-subalgebra of $C(T)$. You can then check that neither $C(T)$ nor $A(D)$ is a $C^*$-algebra with respect to this involution.

\(^2\)A a bounded operator $P$ on a complex Hilbert space $H$ is called a projection if $P = P^* = P^2$. The term orthogonal projection or self-adjoint projection is, perhaps, more accurate. Note that $M = P(H)$ is a closed subspace of $H$ and that $P$ is the usual projection with respect to the direct sum decomposition $H = M \oplus M^\perp$. However, since we are only interested in these sorts of projections, we will settle for the undecorated term “projection.”