Math 116 Numerical PDEs: Homework 5

due Mon 9am, Feb 13

Notice that I refer to "the user" below. When you code, always imagine you are writing a code for an imaginary "user" (which could be your future self), who needs this code to be a simple to use as possible. Oh, and with a documented interface for each function, like I gave in HW2 #4.

1. Set-up for closed curves in the plane. You'll assume the user supplies a polar curve given by a 2π-periodic function $R : \mathbb{R} \to \mathbb{R}^+$, and also supplies its derivative function R'.
   (a) In terms of the functions R and R', make functions of s (inline ones may do) returning the vector $z(s)$, the speed $|z'(s)|$, and the unit-length normal vector $n(s)$.
   (b) Given M, the number of equispaced periodic trapezoid nodes, use the above to fill a $2 \times M$ array of boundary nodes $y_j = z(s_j)$, a similar array of normals n_j, and a $1 \times M$ array of speeds p_j.
   (c) For the choice $R(s) = 1 + 0.3 \cos(3s)$ and $M = 30$, use the above arrays to produce a plot of the boundary nodes as blobs, with each surface normal plotted as a little line with its tail at its own boundary node. [Hint: It should look like a hairy amoeba. Use `axis equal;` so that right angles look like they should. Debug if not.]

2. Here you check that "Gauss' Law" really works, and test numerical convergence.
   (a) Use periodic trapezoidal quadrature with $N = 30$ nodes in the parameter $s \in [0, 2\pi)$, your arrays from question #1, your plotting code from HW4 #5, to write a code which approximates the boundary integral

$$u(x) = \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n_y} ds_y \tag{1}$$

   for a set of points x. (Note this is the double-layer operator, $u = \mathcal{D}\tau$, acting on the density $\tau \equiv 1$.) Use this to make a 3D surface plot of $u(x)$, for x in the square $[2, 2]^2$. Check that interior values are roughly -1, and exterior zero. [To debug, try the circle $R \equiv 1$ first].
   (b) The above plot should approximate -1 in Ω; make a labeled `contourf` plot of $\log_{10}$ of the absolute deviation from this value over the interior of Ω. How does the error seem to vary in the domain?
   (c) For the single fixed location $x = (0.2, 0.1)$, show convergence vs N of error at this point on an appropriate plot, and state the convergence order or rate. What N is needed to reach the minimum error? BONUS: How does the above depend on the choice of point x?

3. Proof of bound on the "far" part in the double-layer jump relation. Fix $y, z \in \partial\Omega$, and let $x = x(h) = z + hn_z$ be a point off the surface for $h \neq 0$. We make a geometric assumption $2h \leq |z - y|$.
   (a) By using the assumption, and using crude but rigorous bounds on things, show that

$$\frac{d}{dh}\left(\frac{\partial\Phi(x,y)}{\partial n_y} - \frac{\partial\Phi(z,y)}{\partial n_y}\right) .$$

   is less than $C/|z - y|^2$ in size, for some fixed number C.

(b) Use this and the mean value theorem in h to show the following. Let $z \in \partial\Omega$, and $r > 0$, then for all $|h| < r/2$,

$$\int_{y\in\partial\Omega, |y-z|\geq r} \left| \frac{\partial\Phi(x,y)}{\partial n_y} - \frac{\partial\Phi(z,y)}{\partial n_y} \right| ds_y \leq C\frac{h}{r^2}$$

for some C which may depend on the domain, but not on z, r or h.

4. At last you solve a BVP, the interior Dirichlet one for Laplace's equation!

(a) Expand your code from #1 to allow the input of the 2nd-derivative function R'', and to produce the curvature function $\kappa(s)$ at the parameter s. [Debug by checking your curvature does what you expect for the curve $R(s) = 1 + 0.3\cos(3s)$.]

(b) Make a routine to fill the $N \times N$ Nyström kernel matrix: diagonal elements via $\kappa(s_j)$, off-diagonal elements via the formula $\frac{\partial\Phi(x,y)}{\partial n_y}$. Don't forget to include the speed function and (constant) weights $2\pi/N$. Oh, and then add the identity. [Hint: debug using `imagesc` on the matrix to see its diagonal entries smoothly match the off-diagonal; the kernel function should be is smooth].

(c) Let's use a BVP with known solution $u(x_1, x_2) = \cos(x_1)e^{x_2}$ which you can check is harmonic. Then its Dirichlet boundary data is $f = u|_{\partial\Omega}$, which you should use to fill a column vector of values $f(s_j)$. Let's fix $N = 30$ nodes. Solve the Nyström linear system corresponding to the integral equation $(I - 2D)\tau = -2f$ to get the τ vector at the nodes. Tweak your code from #2 to compute the solution potential $u^{(N)}(x)$ generated by this τ, at the location $x = (0.2, 0.1)$. [Hint: it should be small!]

BONUS: start investigating convergence vs M, and location x, if you want...