Math 116 Numerical PDEs: Homework 6

due Man 9am, Feb 20

Shorter to let you start projects. Hint: always try to extend/tweak your existing code. First you may want to look over (or fix!) your HW5 codes, rename the routines to informative names, document them, etc.

1. Here your alter your code from HW5 #4 to solve the interior Dirichet BVP for the Helmholtz equation in the same Ω. [Warning: you are now using complex numbers, so to transpose an array you should use `.'` . Also graph-plotting and `imagesc` only handles real-valued arrays.]

   (a) The Helmholtz DLP kernel $\partial\Phi(x,y)/\partial n_y$ is $\Phi(x,y) = (i/4)H_0(k|x-y|)$; note $H_0'(z) = -H_1(z)$. Give your kernel function an extra k input argument and make it switch between Laplace and Helmholtz (which has the same diagonal values) using `if k==0 ... else ... end`.

   (b) Fixing wavenumber $k = 6$, test with boundary data for the solution field $u(x) = 5iH_0(k|x-x_0|)$, with 'source point' $x_0 = (0,2)$ which is outside Ω. Produce a color image of $\log_{10}$ error over the grid $[-1.3, 1.3]^2$ with spacing 0.02, for $N = 50$. [Since special functions are slow, computation will take about 5 sec].

   (c) What now is the convergence order, or rate, at the fixed location $x = (0.2, 0.1)$? Is it as good as for Laplace? What does this suggest about the smoothness of the Helmholtz DLP kernel?

2. Make your code switchable to the *exterior* Helmholtz BVP, which is as simple as changing the signs in the BIE to $(I + 2D)\tau = 2f$. You will now bounce waves off the domain Ω!

   (a) Use this to solve for the scattered field u^s due to the incident plane wave $u^i(x) = e^{ik\hat{n}\cdot x}$ with wavenumber $k = 10$ and direction $\hat{n} = (\cos 0.2, \sin 0.2)$ reflecting from the domain Ω from before with Dirichlet boundary condition. Produce a 2D color image showing $u = u^i + u^s$ over the region $[-4, 4]^2$; this should look familiar from the course webpage. In particular check that $u|_{\partial\Omega}$ heads towards zero.

   BONUS: set the values inside Ω to zero or `NaN` since they are not physically relevant.

   (b) Since the solution is not known analytically, observe the convergence with N until you are confident in the first 4 significant digits of the total field u at the point $(-2, 0)$ and quote them.

3. Here you explore the dependence on wavenumber k of the above Helmholtz kernel matrices—this will be quick since you already have a function for these.

   (a) Plot a graph vs $k \in [0.1, 4]$ of the lowest singular value of the Nyström matrix for $(I - 2D)$ you used for the interior BVP. Locate as accurately as you can the lowest k where its condition number blows up.

   BONUS: use a built-in minimization function (rather than just searching 'by hand'.)

   (b) By choosing generic (i.e. almost any) boundary data f, answer whether the actual physical solution to the interior BVP blows up near this k, or if it is merely a BIE effect?

   (c) Plot a similar graph for the matrix $(I + 2D)$ which you used in the exterior BVP. Locate as accurately as you can the lowest nonzero k where its condition number blows up.

   (d) Answer as before whether the physical solution to the exterior BVP blows up near this k, or if it is merely a BIE effect?

   Note: the k^2 values you found in (a) are the Dirichlet eigenvalues of Ω, and in (c) the Neumann eigenvalues of Ω.