Complex Analysis

References

- *Funktionentheorie*, by Klaus Jänich
- *Complex Analysis*, by Lars Ahlfors
- *Complex Analysis*, by Serge Lang

In the main, I will be following Jänich's book, at least for the beginning part of these lectures.

1 Complex Numbers

An "imaginary" number is introduced, called i (for *imaginary*), which is declared to be a solution of the polynomial equation

$$x^2 + 1 = 0.$$

The field of complex numbers is denoted by $\mathbb{C}$. We have

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}.$$

For $z = x + iy$, we also write $Re(z)$ to denote the real part of z, namely the real number x. Also $Im(z) = y$ is the imaginary part of z.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be complex numbers. Then the addition and multiplication operations are given by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

and

$$z_1 \cdot z_2 = (x_1 \cdot x_2 - y_1 \cdot y_2) + i(x_1 \cdot y_2 + x_2 \cdot y_1).$$

The complex conjugate is $\bar{z} = x - iy$ (that is, $\bar{z} = x + i(-y)$). Therefore $\bar{\bar{z}} = z$. We have $|z|^2 = z\bar{z} = x^2 + y^2$. If $z \neq 0$ (that is, either $x \neq 0$, or $y \neq 0$) then $z\bar{z} > 0$, and we have

$$z^{-1} = \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{z\bar{z}}.$$

If $z \neq 0$ then there are unique real numbers $r > 0$ and $0 \leq \theta < 2\pi$ such that $x = r\cos\theta$ and $y = r\sin\theta$. So let u, $v \in \mathbb{C}$ be non-zero numbers, and let

$$u = r(\cos\theta + i\sin\theta),$$

$$v = s(\cos\psi + i\sin\psi).$$

Then (remembering the rules for combining trigonometric functions), we see that

$$u \cdot v = r \cdot s(\cos(\theta + \psi) + i\sin(\theta + \psi)).$$

If $\mathbb{C}$ is identified with $\mathbb{R}^2$, the 2-dimensional real vector space, then we can identify any complex number $z = x + iy$ with the vector $\begin{pmatrix} x \\ y \end{pmatrix}$. But for any two real numbers x and y, there exists a unique[1] pair of numbers $r \geq 0$, $0 \leq \theta < 2\pi$, with $x = r\cos\theta$ and $y = r\sin\theta$. So let $u = s + it = \begin{pmatrix} s \\ t \end{pmatrix}$ be some other complex number. Then

$$z \cdot u = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \cdot \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix} \cdot \begin{pmatrix} s \\ t \end{pmatrix} = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} s \\ t \end{pmatrix}.$$

[1] Obviously it is not quite unique for the number $z = 0$.

Thus we see that multiplication of complex numbers looks like an (orientation preserving) orthogonal mapping within $\mathbb{R}^2$ — combined with a scalar factor r.

More generally, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an arbitrary linear mapping, represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. How can this mapping be represented in terms of complex arithmetic? We have $f(1) = \begin{pmatrix} a \\ c \end{pmatrix}$ and $f(i) = \begin{pmatrix} b \\ d \end{pmatrix}$. Therefore, using the linearity of f, for $z = x + iy$ we have

$$\begin{aligned} f(z) &= x \cdot f(1) + y \cdot f(i) \\ &= \left(\frac{1}{2}(z + \bar{z})\right) f(1) + \left(\frac{1}{2i}(z - \bar{z})\right) f(i) \\ &= z\left(\frac{1}{2}(f(1) - if(i))\right) + \bar{z}\left(\frac{1}{2}(f(1) + if(i))\right). \end{aligned}$$

Therefore, if $f(1) = -if(i)$, that is, $if(1) = f(i)$, then the mapping is simply complex multiplication. On the other hand, if $f(1) = if(i)$ then we have $f(z) = w \cdot \bar{z}$, where $w = (f(1) + if(i))/2 \in \mathbb{C}$ is some complex number. Writing $w = \begin{pmatrix} s \\ t \end{pmatrix}$, we have

$$f(z) = \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} s & t \\ t & -s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = r \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

for a suitable choice of r and ψ. This is an orientation reversing rotation (again combined with a scalar factor r).

Now let $u = a + ib = \begin{pmatrix} a \\ b \end{pmatrix}$ and $v = c + id = \begin{pmatrix} c \\ d \end{pmatrix}$. What is the scalar product $\langle u, v \rangle$? It is

$$\langle u, v \rangle = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = ac + bd = \mathrm{Re}(u\bar{v}) = \mathrm{Re}(\bar{u}v).$$

Therefore $\langle z, z \rangle = |z|^2$, where $|z| = \sqrt{x^2 + y^2}$ is the absolute value of $z = x + iy$.

2 Analytic Functions

Definition 1. *Any non-empty connected*[2] *open set* $G \subset \mathbb{C}$ *will be called a* region.

So we will generally be interested in functions $f : G \to \mathbb{C}$ defined in regions.

Definition 2. *Let* $f : G \to \mathbb{C}$ *be given, and let* $z_0 \in G$*. If*

$$f'(z_0) = \lim_{\substack{z \to z_0 \\ z \neq z_0 \\ z \in G}} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, then it is the derivative *of* f *at* z_0*. The function* f *will be called* analytic *in* G *if it is defined, and has a continuous derivative everywhere in* G*. The word* holomorphic *is also used, and it is synonymous with the word analytic.*

As in real analysis, we have the simple rules for combining the derivatives of two functions f and g:

$$\begin{aligned} (f + g)' &= f' + g', \\ (f \cdot g)' &= f' \cdot g + f \cdot g', \\ \left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2}, \\ (g \circ f)'(z) &= g'(f(z))f'(z). \end{aligned}$$

[2]Recall that in $\mathbb{R}^2$, every open connected subset is also path-connected.

Nevertheless, there is a very big difference between the idea of a derivative in complex analysis, and the familiar derivative in real analysis. The reason for this is that a *common limit* must exist, regardless of the direction with which we approach the point z_0 in the complex plane. This leads to the Cauchy-Riemann differential equations.

Looking at the definition of the complex derivative, one immediately sees that it is really a special version of the total derivative (as in analysis 2) in $\mathbb{R}^2$. Thus, for $\xi \in \mathbb{C}$ sufficiently small (that is $|\xi|$ small), we have

$$f(z_0 + \xi) = f(z_0) + A\xi + |\xi|\psi(\xi),$$

where A is a 2×2 real matrix, and $\lim_{\xi \to 0} \psi(\xi) = 0$. But what is A? It represents multiplication with the complex number $f'(z_0) = a + ib$, say. That is, $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

So what are these real numbers a and b? Let $f(z) = u(z) + iv(z)$, where $u, v : G \to \mathbb{R}$ are real functions. Then writing $z = x + iy$, we have $f(x + iy) = u(x + iy) + iv(x + iy)$. Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we can consider the partial derivatives of u and v. Since A is simply the Jacobi matrix of the mapping f at the point z_0, we must have

$$A = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Therefore

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are the Cauchy-Riemann equations. Another way to express this is to simply say that we must have

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

Thinking in geometrical terms, we see that if f is analytic, then it is a *conformal* mapping, at least at the points where f' is not zero. That means that, locally, the mapping preserves angles. Looked at up close, the mapping is

$$f(z_0 + \xi) = \underbrace{f(z_0)}_{\text{translation}} + \underbrace{r}_{\text{scalar factor}} \underbrace{\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}}_{\text{rotation}} + \text{Something small.}$$

Of course, as we have already seen, the rotation preserves orientation. Thus it is an element of the group $SL_2(R)$.

Another interesting detail is that the real and imaginary parts of an analytic function are themselves harmonic functions. Anticipating a later conclusion, let us assume that the parts of the analytic function $f = u + iv$ are twice continuously differentiable. Since $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}.$$

Or, expressed in another standard form of notation,

$$u_{xx} + u_{yy} = \Delta u = 0.$$

Here, Δ is the Laplace operator. Similarly, we see that $\Delta v = 0$.

All of this shows that we cannot simply choose any old smooth function $f : G \to \mathbb{C}$ and expect it to be analytic. On the contrary, there is a very great "rigidity", which means that most smooth functions — even though they may be partially differentiable when considered as mappings of 2-dimensional Euclidean space — are not complex differentiable.

Examples

1. The first example is the nice and smooth function $f(z) = f(x + iy) = x^2 + y^2$. Here $u_x = 2x$ and $v_y = 0$. But according to the Cauchy-Riemann equations, we must have $u_x = v_y$; that is, $x = 0$. This only holds along a single line in the complex plane $\mathbb{C}$. Therefore it certainly can't hold in

any region of $\mathbb{C}$ (since regions are defined to be open), and thus f, despite all appearances of being a nice function, is definitely *not* analytic.

2. Having been cautioned by the previous example, let us try to construct an analytic function. For example, let us assume that $u(x+iy) = x$. What possibilities are there for $v(x+iy)$? Since $u_x = 1 = v_y$ and $u_y = 0 = v_x$, it is clear that the only possibility is $v(x+iy) = y + \text{constant}$. So this is just the rather boring function $f(z) = z + \text{constant}$.

3. Thinking more positively, we have just seen that the simplest non-trivial polynomial, namely $f(z) = z$, is analytic throughout $\mathbb{C}$. Of course the simplest polynomial, $f(z) = \text{constant}$, is also analytic. But then, noting that we can use the sum and product rules for differentiation in complex analysis, we see that *any* arbitrary complex polynomial is analytic throughout $\mathbb{C}$. Indeed, z^{-n} is also analytic (in $\mathbb{C} \setminus \{0\}$) for any $n \in \mathbb{N}$.

3 Path Integrals

Let $t_0 < t_1$ be two real numbers. Then a continuous mapping $\gamma : [t_0, t_1] \to G \subset \mathbb{C}$ is a *path* in the region G of $\mathbb{C}$. In the analysis lecture we learned that γ is *rectifiable* if a number L_γ exists such that for all $\epsilon > 0$, a $\delta > 0$ exists such that for every partition $t_0 = a_0 < a_1 < \cdots < a_n = t_1$ which is such that $a_{j+1} - a_j < \delta$ for all j, we have

$$\left| L_\gamma - \sum_{j=1}^{n} |\gamma(a_{j+1}) - \gamma(a_j)| \right| < \epsilon.$$

Let $\gamma(t) = \gamma_r(t) + i\gamma_i(t)$, where γ_r, $\gamma_i : [t_0, t_1] \to \mathbb{R}$ are real-valued functions. Then we say that the path is *continuously differentiable* if both the functions γ_r and γ_i are continuously differentiable. In this case, $\gamma' = \gamma_r' + i\gamma_i'$ is also a path in $\mathbb{C}$.[3] We also learned that continuously differential paths are always rectifiable, and we have

$$L_\gamma = \int_{t_0}^{t_1} |\gamma'(t)| dt.$$

All of this has already been dealt with in the analysis lecture. For us now, the interesting thing is to think about path integrals through a region where a complex-valued function is given.

Definition 3. *Let $G \subset \mathbb{C}$ be a region, and let $f : G \to \mathbb{C}$ be a function. Furthermore, let $\gamma : [t_0, t_1] \to G$ be a differentiable path. Then the path integral of f along γ is*

$$\int_\gamma f(z) dz \stackrel{def}{=} \int_{t_0}^{t_1} f(\gamma(t)) \cdot \gamma'(t) dt,$$

assuming it exists.

The integral here is simply the sum of the integrals over the real and the imaginary parts. It is not necessary to assume that γ is continuously differentiable, but we will assume that it is *piecewise* continuously differentiable. That is, there is a partition of the interval $[t_0, t_1]$ such that it is continuously differentiable along the pieces of the partition. So from now on, we will (almost) always assume that all paths considered are piecewise continuously differentiable.

As an exercise (using the substitution rule for integrals), one sees that the path integral does not depend on the way the path is parameterized. The simplest case is that, say $\gamma(t) = t$. Then (taking t from 0 to 1) we just have $\int_\gamma f(z)dz = \int_0^1 f(t)dt$. Almost equally simple is the case that $\gamma(t) = it$. Then we have $\int_\gamma f(z)dz = i\int_0^1 f(it)dt$.

Increasing the complexity of our thoughts ever so slightly, we arrive at the first version of Cauchy's integral theorem.

[3] Thinking in terms of 2-dimensional real geometry, we can say that $\gamma'(t)$ is the "tangent vector" to $\gamma(t)$.

4 Cauchy's Theorem (simplest version)

Theorem 1. *Let* $G \subset \mathbb{C}$ *be a region, and assume that the function* $f : G \to \mathbb{C}$ *has an antiderivative (auf deutsch:* Stammfunktion*)* $F : G \to \mathbb{C}$ *with* $F' = f$. *Let* γ *be a closed path*[4] *in* G. *(*Closed *means that* $\gamma(t_0) = \gamma(t_1)$.*) Then* $\int_\gamma f(z)dz = 0$.

Proof.

$$\int_\gamma f(z)dz = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)dt = \int_{t_0}^{t_1} (F(\gamma(t)))'dt = F(\gamma(t_1)) - F(\gamma(t_0)) = 0.$$

□

Since every polynomial has an antiderivative, it follows that the path integral around a closed path for any polynomial is zero.

Of course this is all a bit *too* trivial. So let's call the following theorem the simplest version of Cauchy's integral theorem.

Theorem 2. *Let* Q *be a (solid) triangle in the complex plane. Assume that* $Q \subset G \subset \mathbb{C}$*, and take* $f : G \to \mathbb{C}$ *to be an analytic function. Let* γ *be the closed path traveling around the three sides of* Q. *Then* $\int_\gamma f(z)dz = 0$.

Proof. We may assume that γ begins and ends in a corner of Q — for example the "lowest" corner in the complex plane. If the lower side of Q is parallel to the real number axis, then take the right-hand corner on that side. Let us now divide the sides of Q in half, connecting the half-way points with straight line segments, thus creating four equal sub-triangles, $Q_1, \dots, Q_4$. Let γ_j be the path traveling around the boundary of Q_j, for $j = 1, \dots, 4$. Again we may assume that each γ_j begins and ends in the bottom right corner of it's triangle. So we have

$$\int_\gamma f(z)dz = \sum_{j=1}^{4} \int_{\gamma_j} f(z)dz.$$

Assume further that each of these paths is parameterized in the simplest way possible, so that $|\gamma'| = 1$ and $|\gamma_j'| = 1$ for all the j. Therefore L_γ is the sum of the lengths of the three sides of the triangle Q, and $L_{\gamma_j} = L_\gamma/2$ for each of the j.

Let's say that j_1 is one of the numbers between one and four such that the value of

$$\left| \int_{\gamma_{j_1}} f(z)dz \right|$$

is the greatest. Then we certainly have

$$\left| \int_\gamma f(z)dz \right| \le 4 \left| \int_{\gamma_{j_1}} f(z)dz \right|.$$

The next step is to concentrate on the triangle Q_{j_1}. As with Q, we subdivide Q_{j_1} into four equal sub-triangles and we take paths around their boundaries. Choose Q_{j_2} to be one of these sub-triangles of Q_{j_1} which is such that the value of

$$\left| \int_{\gamma_{j_2}} f(z)dz \right|$$

is the greatest. Here γ_{j_2} is the path around the boundary of Q_{j_2}. Now we have $L_{\gamma_{j_2}} = L_\gamma/4$, and

$$\left| \int_\gamma f(z)dz \right| \le 16 \left| \int_{\gamma_{j_2}} f(z)dz \right|.$$

[4] That is, a continuous, closed, piecewise continuously differentiable path.

This whole process is continued indefinitely, so that we obtain a sequence of triangles, becoming smaller and smaller, converging to a point, $z_0 \in Q$ say,

$$Q \supset Q_{j_1} \supset Q_{j_2} \supset \cdots \to z_0 \in Q.$$

For each n we have $L_{\gamma_{j_n}} = L_\gamma/2^n$ and

$$\left|\int_\gamma f(z)dz\right| \le 4^n \left|\int_{\gamma_{j_n}} f(z)dz\right|.$$

But we have assumed that f is analytic, in particular it is differentiable at the point z_0. Thus we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \chi(z),$$

for points z in G, where $\chi : G \to \mathbb{C}$ is a continuous function with

$$\lim_{z\to z_0} \frac{\chi(z)}{z - z_0} = 0.$$

So let $\epsilon > 0$ be arbitrarily given. Then there exists some $\delta > 0$ such that $|\chi(z)| < \epsilon|z - z_0|$ for all z with $0 < |z - z_0| < \delta$.

Now we need only choose n so large that $|z - z_0| < \delta$ for all $z \in Q_{j_n}$. For such z we have

$$|\chi(z)| < \epsilon|z - z_0| < \frac{\epsilon L_\gamma}{2^n}.$$

On the other hand, again since the length of γ_{j_n} is $L_\gamma/2^n$, we have

$$\left|\int_{\gamma_{j_n}} \chi(z)dz\right| < \frac{\epsilon L_\gamma}{2^n} \cdot \frac{L_\gamma}{2^n}.$$

Bearing in mind Theorem 1 (and remembering that $f'(z_0)$ is simply a constant complex number), we conclude that

$$\begin{aligned}\left|\int_\gamma f(z)dz\right| &= \left|\int_\gamma (f(z_0) + f'(z_0)(z - z_0) + \chi(z))dz\right| \\ &= \left|\int_\gamma \chi(z)dz\right| \\ &\le 4^n \left|\int_{\gamma_{j_n}} \chi(z)dz\right| < 4^n \cdot \frac{\epsilon L_\gamma}{2^n} \cdot \frac{L_\gamma}{2^n} = \epsilon L_\gamma^2.\end{aligned}$$

Since ϵ was arbitrary and L_γ remains constant, we conclude that $\int_\gamma f(z)dz = 0$. □

But now Theorem 2 can be turned around, and we obtain (almost) the converse.

Theorem 3. *Assume $G \subset \mathbb{C}$ is a region and $f : G \to \mathbb{C}$ is a continuous function. Assume furthermore that for any solid triangle Q contained in G we have $\int_\gamma f(z)dz = 0$, where γ is the path around the triangle. Then f has an antiderivative in every open disc contained in G. That is, let $U = \{z \in \mathbb{C} : |z - z_*| < r\}$ be some such disc, where z_* is a complex number (the middle point of the disc) and $r > 0$ is the radius of the disc. Then there exists $F : U \to \mathbb{C}$ with $F'(z) = f(z)$ for all $z \in U$.*

Proof. By replacing f with the function f_*, where $f_*(z) = f(z - z_*)$, we obtain the situation that $z_* = 0$. Clearly, if the theorem is true for f_*, then it is also true for f. Therefore, without loss of generality, we may simply assume that $z_* = 0$.

Within U the function F is defined to be

$$F(z) = \int_{\alpha_z} f(w)dw.$$

Here, α_z is the straight line from 0 to z, that is, $\alpha_z(t) = tz$. To show that F really is an antiderivative to f in U, let z_0 be some arbitrary point of U and let z be some other point of U. Let β be the straight line connecting z_0 to z. That is, $\beta(t) = (1-t)z_0 + tz$. Being a triangle, the integral of f around the path from 0 out to z, then from z to z_0 then from z_0 back to 0 must itself be zero. That is,

$$F(z) - F(z_0) = \int_\beta f(w)dw.$$

Looking at the definition of the path integral, we see that

$$\int_\beta f(w)dw = \int_0^1 f(\beta(t))\beta'(t)dt = \int_0^1 f((1-t)z_0 + tz)(z - z_0)dt.$$

Therefore

$$\begin{aligned}\frac{F(z) - F(z_0)}{z - z_0} &= \frac{1}{z - z_0}\int_0^1 f((1-t)z_0 + tz)(z - z_0)dt \\ &= \int_0^1 f((1-t)z_0 + tz)dt\end{aligned}$$

Since f is continuous at the point z_0, we have

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = F'(z_0) = f(z_0).$$

□

Combining this theorem with Theorems 1 and 2, we see that if $D \subset G$ is a closed disc, and γ is the circle of it's boundary, then $\int_\gamma f(z)dz) = 0$, for any analytic function defined in the region G. In fact, if γ is *any* (piecewise continuously differentiable and continuous) closed path contained within this disc-like G, then $\int_\gamma f(z)dz) = 0$. For example we can look at a rectangle $[a, b] \times [c, d]$ contained within G. Since the rectangle can be taken to be a union of two triangles, attached along one side, we see that also the path integral around the rectangle must be zero.

More generally, the following theorem will prove to be useful.

Theorem 4. *Let $Q = \{x + iy : 0 \le x, y \le 1\}$ be the standard unit square in $\mathbb{C}$. Take ζ to be the standard closed path, traveling around the boundary of Q once in a counterclockwise direction, beginning and ending at 0. Assume that a continuously differentiable mapping $\varphi : Q \to \mathbb{C}$ is given, such that $\varphi(Q) \subset G$, a region where an analytic function $f : G \to \mathbb{C}$ is defined. Let $\gamma = \varphi \circ \zeta$ be the image of ζ under φ. Then $\int_\gamma f(z)dz = 0$.*

Proof. Since Q is compact, $\varphi(Q)$ is also compact. Therefore it can be covered by a finite number of open discs in G. But Q can now be partitioned into a finite number of sub-squares $Q_1, \dots, Q_n$ such that $\varphi(Q_j)$ is in each case contained in a single one of these open discs.[5] The theorem then follows by observing that the path integral around each of these sub-square images must be zero. □

A special case is the following.

Theorem 5. *Let D_1 and D_2 be closed discs in $\mathbb{C}$, such that D_2 is contained in the interior of D_1. Let γ_j be the closed path going once, counterclockwise, around the boundary of D_j, $j = 1, 2$. Let $G \subset \mathbb{C}$ be a region containing $D_1 \setminus D_2$ and also containing the boundary of D_2. Assume that $f : G \to \mathbb{C}$ is analytic. Then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$.*

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[5]The inverse images of the open discs in Q form a finite open covering $V_1, \dots, V_m$ of Q. A sequence of partitions of Q can be obtained by cutting it along horizontal and vertical lines spaced $1/n$ apart, for each $n \in \mathbb{N}$. Can it be that for each of these partitions, there exists a sub-square which is not contained completely in one of the open sets V_k? But that would mean that there exists a limit point $q \in Q$ such that for every $\epsilon > 0$, there are infinitely many of these sub-squares contained within a distance of ϵ from q. However $q \in V_k$, for some k, and since V_k is open, there exists an ϵ-neighborhood of q contained entirely within V_k, providing us with the necessary contradiction.

Proof. The annulus between D_2 and D_1 can be taken to be the image of the unit square under a continuously differentiable mapping. (The appropriate picture to illustrate this idea will be given in the lecture!) □

A convenient notation for this situation is the following. Let $\gamma : [0, 1] \to \mathbb{C}$ be the path $\gamma(t) = z_0 + re^{2\pi it}$. Then we simply write

$$\int_\gamma f(z)dz = \int_{|z-z_0|=r} f(z)dz.$$

With this notation, we can say that if the analytic function f is defined in a region containing the annulus $\{z \in \mathbb{C} : r \le |z| \le R\}$, then we must have

$$\int_{|z|=r} f(z)dz = \int_{|z|=R} f(z)dz.$$

5 Cauchy's Integral Formula

Theorem 6. *Let* $G \subset \mathbb{C}$ *be a region and let* $f : G \to \mathbb{C}$ *be analytic. Take* $z_0 \in G$ *and* $r > 0$ *so small that* $\{z \in \mathbb{C} : |z - z_0| \le r\} \subset G$*. Furthermore, let* $|a - z_0| < r$*. Then*

$$f(a) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z-a} dz.$$

Proof. Let $0 < \epsilon < r - |a - z_0|$. According to Theorem 5, we have

$$\int_{|z-z_0|=r} \frac{f(z)}{z-a} dz = \int_{|z-a|=\epsilon} \frac{f(z)}{z-a} dz.$$

But

$$\int_{|z-a|=\epsilon} \frac{f(z)}{z-a} dz = \int_{|z-a|=\epsilon} \frac{f(z)-f(a)}{z-a} dz + \int_{|z-a|=\epsilon} \frac{f(a)}{z-a} dz.$$

Since f is differentiable at the point a, the fraction

$$\frac{f(z)-f(a)}{z-a}$$

converges to the constant number $f'(a)$ in the limit as $\epsilon \to 0$. On the other hand, the path length around the circle, and the tangent vector to this path, approach zero as $\epsilon \to 0$. Thus in the limit, the first integral is zero. As far as the second integral is concerned, we have

$$\int_{|z-a|=\epsilon} \frac{f(a)}{z-a} dz = \int_0^1 \frac{f(a)}{\epsilon e^{2\pi it}} \cdot \epsilon 2\pi i \cdot e^{2\pi it} dt = 2\pi i f(a).$$

Therefore we have

$$\int_{|z-a|=\epsilon} \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

□

A relatively trivial implication is the following theorem.

Theorem 7. *The same assumptions as in Theorem 6. But this time take* z_0 *to be the central point of the circle. Then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})dt.$$

Proof. According to Theorem 6, with $a = z_0$, we have

$$f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z-z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} \cdot rie^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})dt.$$

□

So this is just a kind of “mean value theorem” for analytic functions. It shows quite clearly the difference between real analysis and complex analysis. In real analysis, we can make a smooth change in a function, leaving everything far away unchanged, and the function remains nicely differentiable. But in complex analysis, the precise value of the function is determined by the values on a circle, perhaps far away from the point we are looking at. So a change at one place implies that the whole function must change everywhere in order to remain analytic.

6 Power Series

Definition 4. *A power series is a sum of the form*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where[6] $(a_n)_{n \in \mathbb{N}_0}$ *is some arbitrary sequence of complex numbers and* z_0 *is a given complex number.*

So the question is, for which z does the power series converge? Well it obviously converges for $z = z_0$. But more generally, we can say the following.

Theorem 8. *Let the power series* $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ *be given. Then there exists* $0 \le R \le \infty$, *the* radius of convergence, *such that*

1. *The series is absolutely convergent for* $|z - z_0| < R$, *and uniformly convergent for* $|z - z_0| \le \rho$, *for* $0 \le \rho < R$ *fixed.*
2. *It diverges for* $|z - z_0| > R$.
3. *The radius of convergence is given by* $1/R = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$.
4. *The function given by* $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ *is analytic in the region* $|z - z_0| < R$. *For each such* z, *the derivative is given by the series* $f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$, *and the radius of convergence of this derivative series is also* R.

Proof. Parts 1 and 2 have been proved in the analysis lecture. For 3, let $|z - z_0| < \rho < R$ with $1/R = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$. Thus there exists some $N_0 \in \mathbb{N}$ with $\sqrt[n]{|a_n|} < 1/R$ for all $n \ge N_0$. That is, $|a_n| < 1/R^n$. Therefore

$$|a_n (z - z_0)^n| \le \left| \frac{z - z_0}{R} \right|^n \le \left| \frac{\rho}{R} \right|^n,$$

with $\rho/R < 1$. This is a geometric series which, as is well known, converges. On the other hand, if $|z - z_0| \ge \rho > R$ then there exist arbitrarily large n with $\sqrt[n]{|a_n|} > 1/R$. That is, $|a_n| > 1/R^n$ or

$$|a_n (z - z_0)^n| > \left| \frac{z - z_0}{R} \right|^n > 1.$$

So the series cannot possibly converge, since the terms of the series do not converge to zero.

As far as part 4 is concerned, it is clear that $\lim_{n\to\infty} \sup \sqrt[n]{|n a_n|} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$ since $\lim_{n\to\infty} \sqrt[n]{n} = 1$. So let $f_1(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ be the function which is defined in the region $|z - z_0| < R$. We must show that f is analytic here, with $f' = f_1$. To simplify the notation, let us assume from now on that $z_0 = 0$. Choose some complex number w with $|w| < R$. We must show that the derivative of f exists at w, and it equals $f_1(w)$.

To begin with, we write

$$f(z) = \underbrace{\sum_{k=0}^{n-1} a_k z^k}_{S_n(z)} + \underbrace{\sum_{k=n}^{\infty} a_k z^k}_{T_n(z)},$$

[6] $\mathbb{N}_0$ is the set of non-negative integers.

for $|z| < R$. Therefore we have

$$S'_n(z) = \sum_{k=1}^{n-1} k a_k z^{k-1}.$$

So take some ρ with $|w| < \rho < R$ and we restrict ourselves to examining complex numbers z with $|z| < \rho$. Furthermore, choose $\epsilon > 0$. We must show that there exists a $\delta > 0$ such that if $0 < |z-w| < \delta$ then

$$\left| \frac{f(z) - f(w)}{z - w} - f_1(w) \right| < \epsilon.$$

As a first step, choose N_1 sufficiently large that

$$\left| \frac{T_n(z) - T_n(w)}{z - w} \right| < \frac{\epsilon}{3},$$

for all $n \geq N_1$. This is possible, since

$$\begin{aligned} \left| \frac{T_n(z) - T_n(w)}{z - w} \right| &= \left| \sum_{k=n}^{\infty} a_k \left(\frac{z^k - w^k}{z - w} \right) \right| \\ &= \left| \sum_{k=n}^{\infty} a_k \left(\sum_{j=0}^{k-1} z^j w^{k-j-1} \right) \right| \\ &\leq \sum_{k=n}^{\infty} k a_k \rho^{k-1}. \end{aligned}$$

(Remember that the series is absolutely and uniformly convergent in the closed disc with radius ρ.) Thus for some $N_1 \in \mathbb{N}$, the "tail" of the series beyond N_1 sums to something less than $\epsilon/3$. Similarly, the series defining f_1 is absolutely and uniformly convergent in this disc. Therefore take N_2 to be sufficiently large that

$$|S'_n(w) - f_1(w)| < \frac{\epsilon}{3}$$

for all $n \geq N_2$. Let N be the larger of N_1 and N_2. Finally we must determine the number δ. For this, we note that since S_n is just a polynomial, and thus analytic, we have a $\delta > 0$ such that

$$\left| \frac{S_N(z) - S_N(w)}{z - w} - S'_N(w) \right| < \frac{\epsilon}{3}$$

for all z with $|z - w| < \delta$. In particular, if necessary, we can choose a smaller δ to ensure that such z are in our disc of radius ρ. The fact that

$$\frac{f(z) - f(w)}{z - w} - f_1(w) = \left(\frac{S_N(z) - S_N(w)}{z - w} - S'_N(w) \right) + (S'_N(w) - f_1(w)) + \left(\frac{T_N(z) - T_N(w)}{z - w} \right)$$

shows that

$$\left| \frac{f(z) - f(w)}{z - w} - f_1(w) \right| < \epsilon.$$

□

Theorem 9. *Let $f : G \to \mathbb{C}$ be an analytic function defined in a region G, and let $z_0 \in G$ be given. Then there exists a unique power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ whose radius of convergence is greater than zero, and which converges to $f(z)$ in a neighborhood of z_0.*

Proof. Let $r > 0$ be sufficiently small that $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\} \subset G$. In fact, we will also assume the r is sufficiently small that $z \in G$ for all z with $|z - z_0| = r$. Once again, in order to simplify the notation, we will assume that $z_0 = 0$. That is to say, we will imagine that we are dealing with the function $f(z - z_0)$ rather than the function $f(z)$. But obviously if the theorem is true for this simplified function, then it is also true for the original function. According to theorem 7, for $|z| < r$ we then have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)}{\zeta}\frac{1}{1-\frac{z}{\zeta}}d\zeta \\ &= \frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)}{\zeta}\left(\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^n\right)d\zeta \\ &= \frac{1}{2\pi i}\int_{|\zeta|=r}\left(\sum_{n=0}^{\infty}\frac{f(\zeta)}{\zeta}\left(\frac{z}{\zeta}\right)^n\right)d\zeta \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)}{\zeta^{n+1}}d\zeta\right)z^n \\ &= \sum_{n=0}^{\infty}c_n z^n, \end{aligned}$$

where

$$c_n = \frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)}{\zeta^{n+1}}d\zeta$$

for each n.

Here are a few points to think about in this proof.

- The third equation is true since $|z/\zeta| < 1$, and thus the sum is absolutely convergent.
- The fifth equation is true since the partial sums are uniformly convergent, thus the sum and integral operations can be exchanged.
- Although the function $f(\zeta)/\zeta^{n+1}$ is not differentiable at zero, it is defined and continuous on the (compact) circle $|\zeta| = r$. Thus, although c_n is not always zero, still it is always a well defined complex number, for all n.
- It looks like c_n might vary with r. But this is not the case. Theorem 8 implies that $f^{(n)}(0) = n!c_n$, for all n, and this is certainly independent of r.
- The power series converges to $f(z)$ at *all* points of $B(z_0, r)$.

□

7 Some Standard Theorems of Complex Analysis

Combining the last two theorems, we have:

Corollary (Goursat's Theorem)**.** *The derivative of every analytic function is again analytic. Thus every analytic function has arbitrarily many continuous derivatives.*

We can also complete the statement of theorem 3

Theorem 10 (Morera's Theorem)**.** *Let $G \subset \mathbb{C}$ be a region and let $f : G \to \mathbb{C}$ be continuous such that $\int_\gamma f(z)dz = 0$, for all closed paths which are the boundaries of triangles completely contained within G. Then f is analytic.*

Proof. According to theorem 3, there exists an antiderivative $F : G \to \mathbb{C}$, with $F' = f$. Thus, by Goursat's Theorem, f is also analytic. □

Theorem 11 (Cauchy's estimate for the Taylor coefficients). *Again, let* $f : G \to \mathbb{C}$ *be analytic,* $z_0 \in G$, $r > 0$ *is such that* $D(z_0, r) = \{z : |z - z_0| \leq r\} \subset G$, *and*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

for all $z \in D(z_0, r)$. *Since* f *is continuous and* $D(z_0, r)$ *is compact, we must have* $|f|$ *being bounded in* $D(z_0, r)$. *Let* $M > 0$ *be such that* $|f(z)| \leq M$ *for all* $z \in D(z_0, r)$. *Then we have*

$$|c_n| \leq \frac{M}{r^n}$$

for all n.

Proof.

$$|c_n| = \frac{1}{2\pi}\left|\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}dz\right| = \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f(z)}{(re^{it})^{n+1}} rie^{it}dt\right| \leq \frac{M}{2\pi}\int_0^{2\pi}\frac{dt}{r^n} = \frac{M}{r^n}.$$

□

Definition 5. *Let the function* $f : \mathbb{C} \to \mathbb{C}$ *be defined throughout the whole complex plane, and let it be analytic everywhere. Then we say that* f *is an* entire *function.*

Theorem 12. *A bounded entire function is constant.*

Proof. Assume that the entire function $f : \mathbb{C} \to \mathbb{C}$ is bounded with $|f(z)| \leq M$ say, for all $z \in \mathbb{C}$, where $M > 0$ is fixed. Thus $|c_n| \leq M/r^n$, for all $r > 0$. This can only be true if $c_n = 0$ for all $n > 0$. □

Definition 6. *A field is called* algebraically closed *if every polynomial within the field of degree greater than or equal to one has a root.*

Theorem 13 (The Fundamental Theorem of Algebra). $\mathbb{C}$ *is algebraically closed.*

Proof. Let $f(z) = \sum_{k=0}^{n} a_k z^k$, with $n \geq 1$ and $a_n \neq 0$ be a polynomial of degree n. Looking for a contradiction, we assume that there is *no* root, that is, $f(z) \neq 0$ for all $z \in \mathbb{C}$.

For $z \neq 0$, we have

$$f(z) = z^n\left(a_n + \frac{a_{n-1}}{z} \cdots \frac{a_0}{z^n}\right).$$

Let $L = \max\{|a_{n-1}|, \ldots, |a_0|\}$ and take $R \geq 1$ so large that

$$\left|\frac{a_n}{2}\right| \geq \frac{nL}{R}.$$

That is, $R \geq 2nL/|a_n|$. Then

$$\left|\frac{a_{n-j}}{z^j}\right| \leq \frac{L}{R^j} \leq \frac{L}{R} \leq \frac{|a_n|}{2n}$$

for each j and each $|z| \geq R$. Then we have[7]

$$|f(z)| = |z^n|\left|a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \geq |z^n|\left(|a_n| - n\cdot\frac{|a_n|}{2n}\right) = |z|^n\frac{|a_n|}{2}.$$

Since $|a_n|/2$ remains constant, $|z^n| \cdot |a_n|/2$ becomes arbitrarily large, as $|z| \to \infty$. Therefore $|f(z)| \to \infty$ when $|z| \to \infty$. That is to say, if $M > 0$ is given, then there exists an $r > 0$ such that $|f(z)| > M$ for all z with $|z| > r$. That is, $|1/f(z)| < 1/M$ for $|z| > r$. Now, since $f(z) \neq 0$ always, and f (being a polynomial) is an entire function, we have that $1/f$ is also an entire function. It is bounded outside the closed disc $D(0, r)$, but since the function is continuous, and $D(0, r)$ is compact, it is also bounded on $D(0, r)$. Thus it is bounded throughout $\mathbb{C}$, and is therefore constant, by theorem 12. Therefore, the polynomial f itself is a constant function. This contradicts the assumption that f is of degree greater than zero. □

[7]Note that for a and b arbitrary numbers, we have $|a| = |a + b - b| \leq |a + b| + |b|$ or $|a + b| \geq |a| - |b|$, and more generally, $|a + b_1 + \cdots + b_n| \geq |a| - |b_1| - \cdots - |b_n|$.

8 Zeros of Analytic Functions

Definition 7. *Let* $f : G \to \mathbb{C}$ *be an analytic function defined in a region* G*. A point* $z_0 \in G$ *with* $f(z_0) = 0$ *is called a* zero *of the function.*

So let $z_0 \in G$ be a zero of the analytic function $f : G \to \mathbb{C}$. As usual, without loss of generality, we may assume that $z_0 = 0$. As we have seen, we can choose some $r > 0$ such that $B(0, r) = \{z : |z| < r\} \subset G$ and

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

for all such $z \in B(0, r)$.

The fact that $f(0) = 0$ means that $c_0 = 0$. Let $k > 0$ be the smallest integer such that $c_k \neq 0$. (If $c_n = 0$ for *all* n, then f is simply the constant function which is zero everywhere. This is not what we are interested in here so we will assume that some k exists with $c_k \neq 0$.) The easiest case is then that $k = 1$. In this case, we have

$$f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1},$$

and in particular $f'(0) = c_1 \neq 0$. Could it be that for every $\epsilon > 0$ there exists a complex number z_ϵ with $0 < |z_\epsilon| < \epsilon$ and yet $f(z_\epsilon) = 0$? But that would imply that

$$f'(0) = \lim_{\epsilon \to 0} \frac{f(z_\epsilon) - f(0)}{z_\epsilon - 0} = 0.$$

This is impossible, since $f'(0) \neq 0$. Therefore we have:

Theorem 14. *Let* $f : G \to \mathbb{C}$ *be analytic and let* $z_0 \in G$ *be such that* $f(z_0) = 0$ *while* $f'(z_0) \neq 0$*. Then there exists an* $\epsilon > 0$ *such that* $B(z_0, \epsilon) \subset G$ *and the only zero of* f *in* $B(z_0, \epsilon)$ *is the single number* z_0*.*

Of course, another way of thinking of these things — and remembering what was done in Analysis II — is to consider f to be a continuously differentiable mapping of G into $\mathbb{C}$, represented as $\mathbb{R}^2$. The mapping f is then totally differentiable, and the derivative at z_0 is not singular; thus it is a local bijection around z_0.

More generally, we might have k being greater than 1. In any case, the number k is called the *order* of the zero. A zero of order 1 is also called a *simple zero*.

Theorem 15. *Let* $f : G \to \mathbb{C}$ *be analytic in the region* G*, and let* $z_0 \in G$ *be a zero of* f *of order* k*. Then there exists an* $\epsilon > 0$ *such that in the open disc* $B(z_0, \epsilon)$ *of radius* ϵ *around* z_0 *we have* $f(z) = (h(z))^k$*, where* $h : B(z_0, \epsilon) \to \mathbb{C}$ *is analytic with a simple zero at* z_0*.*

Proof. For sufficiently small $\epsilon > 0$, we can write

$$f(z) = \sum_{n=k}^{\infty} c_n (z - z_0)^n = (z - z_0)^k \left(c_k + \sum_{n=k+1}^{\infty} c_n (z - z_0)^{n-k} \right) = (z - z_0)^k g(z)$$

say, for $z \in B(z_0, \epsilon)$. Here $g : B(z_0, \epsilon) \to \mathbb{C}$ is analytic, and $g(z_0) = c_k \neq 0$. Thus there are k distinct k-th roots of the number $g(z_0) = c_k$.[8] Let z_1 be one of these k-th roots of $g(z_0)$. Now consider the particular polynomial function $\varphi_k(z) = z^k$. We know that φ_k is an entire function, and that $\varphi_k'(z_1) = k z_1^{k-1} \neq 0$, since $z_1 \neq 0$. So there is a neighborhood U_1 of z_1, and a neighborhood V_1 of $g(z_0)$, such that $\varphi_k : U_1 \to V_1$ is a bijection[9], with $\varphi_k(z_1) = g(z_0)$. Let $\varphi_k^{-1} : V_1 \to U_1$ be the inverse mapping. Now choose $\epsilon > 0$ so small that $g(B(z_0, \epsilon)) \subset V_1$. Then take $h(z) = (z - z_0) \cdot \varphi_k^{-1}(g(z))$. This defines a function $h : B(z_0, \epsilon) \to \mathbb{C}$ which satisfies our conditions.[10] □

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[8] For any $w \neq 0$ in $\mathbb{C}$ we have $w = re^{i\theta}$ say. Then each of the numbers $\sqrt[k]{r} \cdot e^{i\theta/k + 2\pi i l/k}$, for $l = 0, \ldots, k-1$, is a different k-th root of w.

[9] This is again Analysis II. φ_k is totally differentiable and non-singular at z_1.

[10] As in real analysis (the proof is the same here in complex analysis) we have the rule that if φ is an invertible differentiable function (with non-vanishing derivative), then φ^{-1} is also differentiable, with derivative $(\varphi^{-1})'(z) = 1/\varphi'(\varphi^{-1}(z))$.

Theorem 16. *Again let* $f : G \to \mathbb{C}$ *be analytic in the region* G*, and let* $z_0 \in G$ *be a zero of order* k*. Then there exists an* $\epsilon_0 > 0$ *and an open neighborhood* $U_{\epsilon_0} \subset G$ *of* z_0 *with* $f(U_{\epsilon_0}) = B(0, \epsilon_0)$*. Within* U_{ϵ_0}*,* z_0 *is the only zero of* f*, and if* $w \neq 0$ *in* $B(0, \epsilon_0)$ *then there are precisely* k *different points* $v_1, \ldots, v_k$ *in* U_{ϵ_0} *with* $f(v_j) = w$*, for all* j*.*

Proof. Since f is continuous and $B(0, \epsilon_0)$ is open, it follows that $U_{\epsilon_0} = f^{-1}(B(0, \epsilon_0))$ is also open, regardless of how the number $\epsilon_0 > 0$ is chosen. So we begin by choosing an $\epsilon_1 > 0$ sufficiently small that we can use theorem 15 and write $f(z) = (h(z))^k$, for all $z \in B(z_0, \epsilon_1)$. Since h has a simple zero at z_0, and therefore the derivative at z_0 is not zero ($h'(z_0) \neq 0$), there exists a neighborhood of z_0 such that h is a bijection when restricted to the neighborhood. So let $\epsilon > 0$ be chosen sufficiently small that $B(0, \epsilon)$ is contained within the corresponding neighborhood of 0. Finally, with this ϵ, we take $\epsilon_0 = \epsilon^k$. Then if $w \neq 0$ in $B(0, \epsilon_0)$, we have k different k-th roots of w, lets call them $u_1, \ldots, u_k$. They are all in $B(0, \epsilon)$. Therefore each has a unique inverse under h, namely $v_j = h^{-1}(u_j)$, for $j = 1, \ldots, k$. Is it possible that some other point, v say, not equal to any of the v_j, also is such that $f(v) = (h(v))^k = w$? But then $h(v)$ would also be a k-th root of w, not equal to any of the u_j, since after all, h^{-1} is a bijection when restricted to $B(z_0, \epsilon_0)$. This is impossible, owing to the fact that there are only k different k-th roots of w. □

9 Simple Consequences

Theorem 17. *Assume* $f, g : G \to \mathbb{C}$ *are two analytic functions defined on a region* G *such that the set* $\{z \in G : f(z) = g(z)\}$ *has an accumulation point. Then* $f = g$*.*

Proof. Let $z_0 \in G$ be such an accumulation point. Then z_0 is a zero of the analytic function $f - g$. But this is not an isolated zero. Therefore $f - g = 0$, the trivial constant function. □

Theorem 18. *Again,* $f : G \to \mathbb{C}$ *is analytic and it is not a constant function. Then* $f(G)$ *is also a region (that is, open and connected) in* $\mathbb{C}$*.*

Proof. Since f is continuous, $f(G)$ must be connected. Is $f(G)$ open? Take $w_0 \in f(G)$, and $z_0 \in G$ with $f(z_0) = w_0$. So then z_0 is a zero of the analytic function $f - w_0$. Since $f - w_0$ is not constant, it follows that z_0 is a zero of some particular finite order. Theorem 16 now shows that w_0 lies in the interior of $f(G)$. □

Theorem 19. *Let* $f : G \to \mathbb{C}$ *be analytic and not constant. (*G *is a region.) Let* $z_0 \in G$*. Then there exists another point* $z_1 \in G$ *with* $|f(z_1)| > |f(z_0)|$*.*

Proof. For otherwise, $f(z_0)$ would lie on the boundary of $f(G)$ in $\mathbb{C}$, and thus $f(G)$ would not be open in contradiction to theorem 18. □

Theorem 20 (The Lemma of Schwarz)**.** *Let* $D = \{z \in \mathbb{C} : |z| \leq 1\}$ *be the closed unit disc in* $\mathbb{C}$*. Assume* $D \subset G$*, a region in* $\mathbb{C}$*, and* $f : G \to \mathbb{C}$ *is analytic with* $f(D) \subset D$ *and* $f(0) = 0$*. Then* $|f'(0)| \leq 1$ *as well, and in fact* $|f(z)| \leq |z|$ *for all* $z \in D$*. If either* $|f'(0)| = 1$ *or there exists some* z_0 *with* $0 < |z_0| < 1$ *such that* $|f(z_0)| = |z_0|$*, then we must have* f *being a simple rotation. i.e.* $f(z) = e^{i\theta} \cdot z$*, for some* θ*.*

Proof. Since $f(0) = 0$, we can write

$$f(z) = z \cdot \left(\sum_{n=1}^{\infty} c_n z^{n-1} \right) = z \cdot g(z)$$

say, where g is an analytic function, defined in some neighborhood of D. So $f'(0) = g(0)$ and $|f(z)| = |z| \cdot |g(z)| \leq 1$. Thus $|g(z)| \leq 1/|z|$. This holds in particular for $|z| = 1$.

On the other hand, theorem 18 says that $g(B(0, 1)) \subset \mathbb{C}$ is open in $\mathbb{C}$.[11] If there were some point $z_* \in B(0, 1)$ with $|g(z_*)| > 1$ then we could choose it to be a point such that this value is maximal. However that would then be a boundary point of $f(B(0, 1))$, contradicting the fact that $f(B(0, 1))$ is open. Thus $|g(z)| \leq 1$ for all $z \in D$. In particular, $|f'(0)| = |g(0)| \leq 1$ and $|f(z)| = |z| \cdot |g(z)| \leq |z|$ for all $z \in D$.

[11] Here $B(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$ is the *open* disc centered at zero, with radius 1.

Finally, let us assume that a z_0 exists with $0 < |z_0| < 1$ such that $|f(z_0)| = |z_0|$. That means $|g(z_0)| = 1$. But according to theorem 19, this can only be true if g is a constant function. Since it is a constant number with absolute value 1, it must be of the form $e^{i\theta}$, for some θ. □

10 Analytic Continuation

We now know that an analytic function $f : G \to \mathbb{C}$ can be represented as a power series centered at any given point $z_0 \in G$. The function f is equal to the function defined by the power series in the largest possible open disc around z_0 which is contained in G. But of course G is not, in general itself an open disc. Therefore there might be parts of G where f is not given by this power series centered on z_0. Or, (thinking about the logarithm function) we might have the situation that G could be expanded to a larger region G_*, with $G \subset G_*$ where the function could be defined. But then perhaps there might be different ways of "continuing" this definition of f from G to G_*.

Therefore let us consider a chain of open discs $(B_1, \ldots, B_n)$ say, with

$$B_j = \{z \in \mathbb{C} : |z - p_j| < r_j\}$$

for a chain of points p_j which are the centers of the discs, and numbers $r_j > 0$, which are the radiuses in each case. We assume that it is a connected chain in the sense that $B_j \cap B_{j+1} \neq \emptyset$, in each case. For each B_j let us assume that an analytic function $f_j : B_j \to \mathbb{C}$ exists, such that in the region of overlap, we have $f_j(z) = f_{j+1}(z)$ for $z \in B_j \cap B_{j+1}$.

Definition 8. *The functions* f_j *here are called* function elements, *and if we have* $f_j(z) = f_{j+1}(z)$ *for* $z \in B_j \cap B_{j+1}$ *for all* $j = 1, \ldots, n-1$, *then we have an* analytic continuation *of the function elements through the chain of open discs. Or, if we consider the ordering of the discs, we can say that the final function* f_n *is obtained by analytic continuation of the initial function* f_1 *through the chain of discs.*

Theorem 17 shows that if, say f_1 is given in B_1, and there exists a chain of open discs allowing some analytic continuation, then this continuation is *unique*.

Theorem 21. *Let* $(B_1, \ldots, B_n)$ *be a chain of open discs with* $B_j \cap B_{j+1} \neq \emptyset$, *for* $j = 1, \ldots, n-1$. *Let* $f_1 : B_1 \to \mathbb{C}$ *be some given analytic function. Then there exists an analytic continuation of* f_1 *throughout the chain*[12] *if and only if there also exists an analytic continuation of* $f_1' : B_1 \to \mathbb{C}$ *(the derivative of* f_1*) throughout the chain.*

Proof. "$\Rightarrow$" is trivial. (Just take f_j', the derivative of f_j, for each j.)

As far as "$\Leftarrow$" is concerned, we are assuming that there is an analytic continuation of the function f_1'. To avoid confusion, let us call this function $g_1 : B_1 \to \mathbb{C}$. i.e. $g_1(z) = f_1'(z)$, for all $z \in B_1$. The assumption is that for each j there is an analytic function $g_j : B_j \to \mathbb{C}$, providing an analytic continuation, starting with g_1. We now use induction on the number n. For $n = 1$ there is nothing to prove. So let $n > 1$, and assume that we have an analytic continuation of g_1 along the chain of open discs from B_1 to B_{n-1}, giving $g_j : B_j \to \mathbb{C}$ such that $g_j = f_j'$, for each $j < n$. According to theorem 3, g_n has an antiderivative, $G_n : B_n \to \mathbb{C}$, with $G_n' = g_n$. Now in the region $B_{n-1} \cap B_n$ we have $g_{n-1} = g_n$. That is, $f_{n-1}' = G_n'$, or $f_{n-1}' - G_n' = 0$. Thus $f_{n-1} - G_n = k$ say, where k is a constant number. But then the function $G_n + k$ is also an antiderivative to g_n, and we can take $f_n = G_n + k$. □

One way to think about these chains of discs is to imagine that they are associated with a path, namely a path starting at p_1 then following a straight line to p_2, then a straight line to p_3, and so forth, finally ending at p_n. So each straight segment, say from p_j to p_{j+1} is contained in the union of the two discs $B_j \cup B_{j+1}$.

Let's generalize this idea in the following way. Let $\gamma : [t_0, t_1] \to \mathbb{C}$ be a continuous path. (It doesn't have to be differentiable here.) Let $t_0 = \tau_0 < \tau_1 < \cdots < \tau_n = t_1$ be a partition of the interval $[t_0, t_1]$. Assume that we have a corresponding set of open discs $B_j = \{z \in \mathbb{C} : |z - \gamma(\tau_j)| < r_j\}$ (where $r_j > 0$) such that $\gamma(t) \in B_j \cup B_{j+1}$, for $t \in [\tau_j, \tau_{j+1}]$, for all relevant j. Then we will say that we have a chain of discs along the path γ. Furthermore, if we have a sequence of analytic function elements giving an

[12] That is to say, there exists a set of function elements forming an analytic continuation, such that the first element in the chain of function elements is f_1.

analytic continuation along this chain, then we will say that the function is analytically continued along the path. The next theorem shows that this is a property of the path, independent of the choice of discs along the path.

Theorem 22. *Let* $\gamma : [t_0, t_1] \to \mathbb{C}$ *be continuous and let* $B = \{z \in \mathbb{C} : |z - \gamma(t_0)| < r\}$ *and* $B_* = \{z \in \mathbb{C} : |z - \gamma(t_1)| < r_*\}$ *(with both* r, $r_* > 0$*) be open discs centered on* $\gamma(t_0)$ *and* $\gamma(t_1)$*, respectively. Let* $t_0 = \tau_0 < \tau_1 < \cdots < \tau_n = t_1$ *and* $t_0 = \rho_0 < \rho_1 < \cdots < \rho_m = t_1$ *be two different partitions of the interval* $[t_0, t_1]$*, giving two different chains of discs along the path, satisfying the conditions listed above, where the first and last discs are* B *and* B_**. Assume that* $f : B \to \mathbb{C}$ *is an analytic function at the first disc and it has an analytic continuation with respect to the first chain of discs, finally giving the analytic function* $g : B_* \to \mathbb{C}$*. Then* f *also has an analytic continuation with respect to the second chain of discs, and it also gives the same function* $g : B_* \to \mathbb{C}$*.*

Proof. If the whole path γ is completely contained within the disc B then, using theorem 9 we see that the analytic continuation is simply given by the power series representing the function.

Therefore let $M \subset [t_0, t_1]$ be defined to be the set of $t_* \in [t_0, t_1]$ such that the theorem is true for the interval $[t_0, t_*]$ (with respect to this path γ). But if $t \in M$, then since γ is continuous, and since the analytic function which has been continued out to the point $\gamma(t_*)$ has a power series representation in a neighborhood of $\gamma(t_*)$, we must have some $\epsilon > 0$ such that $\{t \in [t_0, t_1] : |t - t_*| < \epsilon\} \subset M$. Therefore M is an open subset of $[t_0, t_1]$. If $[t_0, t_1] \setminus M \neq \emptyset$, then the same argument shows that M is closed in $[t_0, t_1]$. Yet the interval $[t_0, t_1]$ is connected. Therefore $M = [t_0, t_1]$. □

All of these thoughts allow us to perform path integrals along continuous, but not necessarily differentiable paths. To see this, take γ to be some continuous path, allowing an analytic continuation from an open disc centered on the starting point of the path. The discs in the finite chain of open discs describing the analytic continuation have centers at various points along the path γ. But now take $\tilde{\gamma}$ to be the piecewise linear path connecting those centers. This gives a path integral, namely $\int_{\tilde{\gamma}} f(z)dz$, where f consists of the function elements along the path. We can now simply *define* $\int_{\gamma} f(z)dz$ to be this integral along $\tilde{\gamma}$. Theorem 22 then shows that the path integral for γ is well defined.

11 The Monodromy Theorem

Why confine our thoughts on analytic continuation to a single continuous path γ? After all, it seems obvious that we can move the path back and forth to some extent — at least when staying within the open discs which we are using — without affecting the arguments of the previous section. This is the idea of the Monodromy Theorem. But first we must define what "moving a path" is supposed to mean.

Definition 9. *Let* $Q = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}$ *be the unit square, and let* $H : Q \to \mathbb{C}$ *be a continuous mapping such that*

- $H(0, y) = H(0, 0)$*, for all* $y \in [0, 1]$*,*
- $H(1, y) = H(1, 0)$*, for all* $y \in [0, 1]$*,*

Let $\alpha : [0, 1] \to \mathbb{C}$ *be the path* $\alpha(t) = H(t, 0)$ *and let* $\beta(t) = H(t, 1)$*. Then we have* $\alpha(0) = \beta(0)$ *and* $\alpha(1) = \beta(1)$*. The mapping* H *is said to be a* homotopy *from* α *to* β*. One also says that* α *and* β *are* homotopic *to one another. If* $G \subset \mathbb{C}$ *is a region, and* $H(Q) \subset G$*, then it is a homotopy* within *the region.*

In the more general setting of topology, this idea of homotopy is quite important. But historically, the idea grew out of these applications in complex analysis. If we work with closed paths α, so that $\alpha(0) = \alpha(1)$, then we can define the *fundamental group* of the topological space. This is dealt with to a greater or lesser degree in all of our textbooks. But I will skip over these things here. Of course, as an additional remark, you should note that the fact that the paths are being parameterized using the unit interval $[0, 1]$ represents no loss of generality. It all works just as well if we use some other interval $[t_0, t_1]$. Finally, note that if $f : G \to \mathbb{C}$ is given as an analytic function and α and β are homotopic to one another within G, then theorem 4 shows that

$$\int_\alpha f(z)dz = \int_\beta f(z)dz.$$

Theorem 23 (Monodromy Theorem)**.** *Let α and β be two homotopic paths in $\mathbb{C}$. Assume there is an open disc B_0 centered on $\alpha(0) = \beta(0)$ and an analytic function $f_0 : B_0 \to \mathbb{C}$. For each $\tau \in [0, 1]$ let $h_\tau : [0, 1] \to \mathbb{C}$ be the path $h_\tau(t) = H(t, \tau)$. (Thus $\alpha = h_0$ and $\beta = h_1$.) Assume that f_0 has an analytic continuation along the path h_τ for each τ. In particular there exists an open disc B_1 centered at $\alpha(1) = \beta(1)$ such that the analytic continuation along α produces the function $f_1 : B_1 \to \mathbb{C}$ and the analytic continuation along β produces the function $\widetilde{f_1} : B_1 \to \mathbb{C}$. Then $f_1 = \widetilde{f_1}$.*

Proof. The proof uses the technique which we have seen in theorem 4. The construction of an analytic continuation along each of the paths h_τ involves some finite chain of open discs. So the set of all such discs covers the compact set $H(Q)$. Take a finite sub-covering. Take the inverse images of the sets in this sub-covering. We obtain a finite covering of Q by open sets. Take a subdivision of Q into sub-squares of length $1/n$, for n sufficiently large, so that each of the sub-squares is contained in a single one of these open sets covering Q. Then let $\gamma_j = h_{j/n}$, for $j = 0, \dots, n$. Now the argument in the proof of the previous theorem (theorem 22) shows that the analytic continuation along γ_j leads to the same function as that along γ_{j+1}, for each relevant j. In particular, the function is uniquely defined through the power series representation in each of the sub-squares. Therefore it's values along one segment of the curve γ_j determines uniquely it's values along the corresponding segment of the next curve γ_{j+1}. Since the endpoints of all of the curves are identical (that is, $\gamma_j(1) = \alpha(1) = \beta(1)$ for all j), we must have the power series expression at this endpont for each of the analytic continuations of the original function being the same. □

12 The Index of a Point With Respect to a Closed Path

Let $\gamma : [t_0, t_1] \to \mathbb{C}$ be a continuous closed curve. Let $z_0 \in \mathbb{C}$ be a point not on the path. Then, as we have seen in Exercise 2.2, we can define a continuous path $\theta : [t_0, t_1] \to \mathbb{R}$ with $\theta(t_0) = 0$ and

$$\frac{\gamma(t) - z_0}{|\gamma(t) - z_0|} = e^{2\pi i \theta(t)}$$

for all $t \in [t_0, t_1]$.

Definition 10. *The number $\theta(t_1) \in \mathbb{Z}$ is called the* index *of the point z_0 with respect to the closed path γ. Sometimes it is also called the* winding number *of γ with respect to z_0. It is denoted $\nu_\gamma(z_0)$*

Theorem 24. *Let α and β be homotopic closed paths, homotopic by a homotopy $H : Q \to \mathbb{C}$. Assume that $z_0 \notin H(Q)$. then $\nu_\alpha(z_0) = \nu_\beta(z_0)$.*

Proof. Let θ_α be the corresponding path in $\mathbb{R}$ (corresponding to α) and let θ_β be the path in $\mathbb{R}$ corresponding to β. Then the homotopy from α to β induces a homotopy of the path θ_α to θ_β in $\mathbb{R}$. Since the endpoints thus remain fixed, the index remains unchanged. □

Theorem 25. *Let γ, z_0 and $\nu_\gamma(z_0)$ be as in the definition (but this time we assume that γ is continuously differentiable). Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0} = \nu_\gamma(z_0).$$

Proof. Obviously the function $1/(z - z_0)$ is analytic in $\mathbb{C} \setminus \{z_0\}$. For simplicity, assume that $z_0 = 0$ and $\gamma(0) = \gamma(1) = 1 = e^{i\cdot 0}$, where $\gamma : [0, 1] \to \mathbb{C} \setminus \{0\}$. Write $\gamma(t) = r(t)e^{2\pi i\theta(t)}$. Let

$$h_\tau(t) = e^{2\pi i\theta(t)}(t + (1 - t)r(t)),$$

for τ, $t \in [0,1]$. This defines a homotopy from γ to a new path $\tilde{\gamma}$, which lies on the unit circle. But

$$\begin{aligned}\int_{\tilde{\gamma}} \frac{dz}{z} &= \int_0^1 \frac{1}{e^{2\pi i\theta(t)}} 2\pi i\theta'(t)e^{2\pi i\theta(t)}dt \\ &= 2\pi i\int_0^1 \theta'(t)dt \\ &= 2\pi i(\theta(1)-\theta(0)) = 2\pi i\nu_\gamma(0).\end{aligned}$$

□

Theorem 26 (Cauchy's Theorem: Complicated version). *Let* $f : G \to \mathbb{C}$ *be an analytic function defined in a region* G *of* $\mathbb{C}$. *Let* $\gamma : [0,1] \to G$ *be a continuous closed path in* G *such that* $\nu_\gamma(a) = 0$ *for all* $a \notin G$. *Then* $\int_\alpha f(z)dz = 0$.

Proof. Since the (image of the) path γ is a compact subset of G, there exists an $\epsilon > 0$ such that for all $t \in [0,1]$, we have $B(\gamma(t),\epsilon) \subset G$. Let $n \in \mathbb{N}$ be sufficiently large that $1/n < \epsilon/\sqrt{2}$. Then split up the whole complex plane $\mathbb{C}$ into a system of small squares of the form

$$Q(p,q) = \left\{x + iy \in \mathbb{C} : \frac{p}{n} \le x \le \frac{p+1}{n}, \frac{q}{n} \le y \le \frac{q+1}{n}\right\},$$

where p, $q \in \mathbb{Z}$. γ meets only finitely many of these small squares, and each of the squares which γ does meet is completely contained in G. Our proof now consists in altering γ, one step after the next, through processes which can be achieved using a homotopy. In the end we get a version of γ which is so simple that the theorem becomes obvious.

- Let Q be one of the squares which γ meets. If γ only runs along the boundary of Q without entering it's interior, then there is nothing further to do in this step of the proof. On the other hand, if γ does enter the interior of Q, then we perform a homotopy on it, moving it to the boundary of Q, but leaving all points of γ which are not in the interior of Q unchanged. This is explained more fully in the lecture. Basically, we take an interior point of Q which is not a point of the path, then we push the part of γ which is in the interior of Q radially from that point out to the boundary. Do this with all the Q's which are on the path, then finally, if necessary, move the endpoint $\gamma(0) = \gamma(1)$ to a vertex of one of these squares.[13] The end result of all this is a homotopy, moving γ so that at the end of the movement it is contained within the lattice of vertical and horizontal lines which make up the boundaries of all the small squares. For simplicity, let us again call this "simplified" version of the path γ.

- Now it may be that this simplified γ is still too complicated. Looking at each individual segment of the lattice, it may be that during a traverse of some particular segment, we see that γ moves back and forth in some irregular way. So again using a homotopy, we can replace γ with a new version which traverses each segment of the lattice in a simple linear path.

- We are still not completely happy with the version of γ which has been obtained, for it might be that γ has a winding number which is not zero with respect to one of the squares Q. That is, let q be an interior point of Q. What is $\nu_\gamma(q)$? If it is not zero, then we can alter γ using a homotopy, changing the winding number with respect to this particular square to zero and not altering the winding number with respect to any of the other squares. Again this is illustrated in the lecture. The idea is simply to add a bit on to the end of γ, going out to a corner of the offending square along the lattice, then around the square a sufficient number of times to reduce the winding number to zero, returning to that corner, then returning back along the same path through the lattice to the starting point. Since γ is compact, there can be at most finitely many such offending squares. At the end of this operation, we have ensured that γ has winding number zero with respect to all points of $\mathbb{C}$ which do not lie on the path γ.

Our curve is now sufficiently well simplified for our purposes. The reason for this is that for every segment of the lattice which γ traverses, it must be that it traverses the segment the same number

[13] Alternatively, and without loss of generality, we may assume that the endpoint of γ is already in a corner of the lattice.

of times in the one direction as it does in the other direction. To see this, let S be some segment of the lattice. To visualize the situation, imagine that it is a vertical segment. Assume that γ traverses the segment u times in the upwards direction, and d times in the downwards direction. Now S adjoins two squares, say Q_1 on the lefthand side, and Q_2 on the righthand side. Let us now alter γ, moving the downwards moving parts which traverse S leftwards across Q_1 to the other three sides of Q_1. On the other hand, the upwards moving parts of γ are moved rightwards across Q_2 to the other three sides of Q_2. Let z_1 be the middle point of the square Q_1, and let z_2 be the middle point of Q_2. Before this movement, the winding number of γ with respect to both points was zero. However, afterwards, $\nu_\gamma(z_1) = a$ and $\nu_\gamma(z_2) = b$. Yet they are in the same region of $\mathbb{C} \setminus \gamma$. Thus the points must have the same winding numbers and so $a = b$.

Finally, after all this fiddling, we see that we must have $\int_\gamma f(z)dz = 0$ since the total of the contributions from the path integrals along each of the segments of the lattice adds up to zero in each case. □

Up to now we have always been performing path integrals around single closed paths. This is quite sensible. But sometimes it is also convenient to imagine two or more closed paths. So let say $\gamma_1, \ldots, \gamma_n$ be n paths, each of which are piecewise continuously differentiable and closed in some region G. We can think of them together, and call them a *cycle*, denoted by the letter Ω say.[14] Then if $f : G \to \mathbb{C}$ is a function, we might be able to perform the path integrals over each of the γ_j seperately, and thus we can define the integral over the whole cycle to be simply the sum of the seperate integrals.

$$\int_\Omega f(z)dz \stackrel{\text{def}}{=} \int_{\gamma_1} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz.$$

Furthermore, if Ω is a cycle and z_0 is a point not on any path of the cycle, then we can define the index of z_0 with respect to the cycle to be the sum of the indices of z_0 with respect to the individual closed paths in the cycle. On the other hand, if we want, we can connect the endpoints of the paths in a cycle together to make a single larger closed path (this is illustrated in the lecture), thus showing that theorem 26 is also true for cycles.

Theorem 27. *Again, let $G \subset \mathbb{C}$ be a region, $f : G \to \mathbb{C}$ be analytic, γ a closed curve in G with winding number zero with respect to all points of $\mathbb{C}$ not in G. Let $z_0 \in G$ be a point which does not lie on the path γ. then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z - z_0} = \nu_\gamma(z_0)f(z_0).$$

Proof. This is a straight-forward generalization of theorem 6. For $r > 0$ sufficiently small, we have

$$\frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)dz}{z - z_0} = f(z_0).$$

Let $\beta(t) = z_0 + r \cdot e^{2\pi i t}$ be the path we are thinking about in this path-integral. (Here $t \in [0, 1]$.) Obviously we have $\nu_\beta(z_0) = 1$. Now take the cycle consisting of the given path γ, together with $-\nu_\gamma(z_0)$ copies of the path β. (If this is a negative number, then we should travel around β in the reverse direction.) We assume that r is so small that the closed disc $D(z_0, r)$ is contained in G and furthermore the path γ does not meet $D(z_0, r)$. Then z_0 has index zero with respect to the cycle consisting of $\gamma - \nu_\gamma(z_0)\beta$. Thus, according to theorem 26,

$$\frac{1}{2\pi i}\int_\Omega \frac{f(z)dz}{z - z_0} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z - z_0} - \nu_\gamma(z_0)\frac{1}{2\pi i}\int_\beta \frac{f(z)dz}{z - z_0} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z - z_0} - \nu_\gamma(z_0)f(z_0) = 0.$$

□

[14] More generally, we might allow paths which are not necessarily closed. In this case one speaks of "chains", but we will not persue this idea further in this lecture.

13 Weierstrass's Convergence Theorem

This is the analog of the theorem in real analysis which states that a sequence of continuous functions which is uniformly convergent converges to a continuous function. But here we are concerned with analytic functions.

Theorem 28 (Weierstrass). *Let $G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$ be an increasing sequence of regions in $\mathbb{C}$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of analytic functions $f_n : G_n \to \mathbb{C}$ for each n. Let $G = \cup_{n=1}^{\infty} G_n$, and assume that in every compact subset of G, the sequence (f_n) converges uniformly. Let $f : G \to \mathbb{C}$ be defined by $f(z) = \lim_{n\to\infty} f_n(z)$ for all $z \in G$. Then f is analytic on G, and furthermore $f_n' \to f'$ uniformly on every compact subset of G.*

Proof. Let $z_0 \in G$ and take $r > 0$ such that $\overline{B(z_0, r)} \subset G$. (That is the closure of the open set $B(z_0, r)$.) Let $N \in \mathbb{N}$ be sufficiently large that $\overline{B(z_0, r)} \subset \cup_{n=1}^{N} G_n$. Then $\overline{B(z_0, r)} \subset G_m$, for all $m \geq N$. According to theorem 2 we have $\int_\gamma f_n(z)dz = 0$ for all triangles in $B(z_0, r)$. Moreover we have

$$\int_\gamma f(z)dz = \lim_{n\to\infty} \int_\gamma f_n(z)dz = 0,$$

owing to the uniform convergence of the sequence. Therefore, by theorem 10, f is analytic. By Goursat's theorem, the derivatives are also analytic. Specifically, let ζ be the boundary of the disc $B(z_0, r)$. Then, using Cauchy's formula (theorem 6), we have

$$f_n'(z_0) = \frac{1}{2\pi i} \int_\zeta \frac{f_n(z)dz}{(z - z_0)^2}.$$

Thus

$$\lim_{n\to\infty} f_n'(z_0) = \frac{1}{2\pi i} \int_\zeta \frac{f(z)dz}{(z - z_0)^2} = f'(z_0),$$

and the fact that the convergence of the f_n is uniform in $\overline{B(z_0, r)}$ shows that $f_n' \to f'$ uniformly. □

An interesting example of this is Riemann's Zeta function. Let $z = x + iy$ with $x > 1$. Then

$$\left| \sum_{n=1}^{\infty} n^{-z} \right| \leq \sum_{n=1}^{\infty} \left| n^{-x} n^{-iy} \right| = \sum_{n=1}^{\infty} n^{-x} \left| e^{-iy \log n} \right| = \sum_{n=1}^{\infty} n^{-x}.$$

Thus the infinte sum $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ defines a function which is the limit of a uniformly convergent sequence of analytic functions (the partial sums) for all z with $\mathrm{Re}(z) > a$, where $a > 1$ is a given constant. Therefore Weierstrass's convergence theorem implies that the Zeta function $\zeta(z)$ is analytic in this region. As a matter of fact, Riemann showed that, with the exception of the obvious isolated singularity at the point $z = 1$, the zeta function can be analytically continued throughout the whole complex plane. The big question is "Where are the zeros of the zeta function?" Some of them are located at negative even integers. (These are the so-called "trivial zeros".) They are not particularly interesting. But there are lots along the vertical line $\mathrm{Re}(z) = 1/2$. The famous Riemann conjecture — which is certainly the greatest unsolved problem in mathematics today — is that *all* of the zeros (apart from the trivial ones) lie on this line.[15]

14 Isolated Singularities

A "singularity" is really nothing more than a point of the complex plane where a given function is not definied. That is, if f is defined in a region $G \subset \mathbb{C}$, then any point $a \notin G$ can be thought of as being a singularity of the function. But this is not really what we are thinking about when we speak of singularities. As an example of what we are thinking about, consider the particular function

$$f(z) = \frac{1}{z}.$$

[15] An American businessman, Mr. Landon T. Clay has put aside one million American dollars each for the solution of a certain collection of outstanding problems in mathematics. The Riemann conjecture is one of them.

Here, except for the special point $z_0 = 0$ which is not so nice, we have a good example of an analytic function. Since only a single point is causing problems with this function, let us more or less ignore this point and call it an *isolated singularity*. So in general, an isolated singularity is a point $z_0 \in \mathbb{C}$ such that $z_0 \notin G$, yet there exists some $r > 0$ with $B(z_0, r) \setminus \{z_0\} \subset G$.

Definition 11. *Let $f : G \to \mathbb{C}$ be analytic, but with an isolated singularity at $z_0 \in \mathbb{C}$. The* residue *of f at z_0 is the number*

$$\text{Res}_{z_0} f(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{|z-z_0|=r} f(z)dz.$$

According to Cauchy's theorem, the residue is a well-defined number, independent of the radius r of the circle of the path used to define it, as long as r is small enough to satisfy our condition.

We identify three different kinds of isolated singularities:

- removable singularities,
- poles,
- essential singularities.

Let's begin with removable singularities. Let $f : G \to \mathbb{C}$ have a singularity at z_0. This singularity is *removable* if it is possible to find some number w_0 such that if we simply define $f_0 : G \cup \{z_0\} \to \mathbb{C}$ by

$$f_0(z) = \begin{cases} f(z), & \text{if } z \in G \\ w_0, & z = z_0 \end{cases}$$

then f_0 is analytic (thus also differentiable at the special point z_o). We see then that a removable singularity is really nothing special. We have simply "forgotten" to put in the correct value of the function f at the isolated point z_0. By putting in the correct value, the singularity disappears.

A *pole* is somewhat more interesting. For example the function

$$f(z) = \frac{1}{z}$$

has a "simple" pole at the point $z_0 = 0$. But then if we multiply f with the "simple" polynomial $g(z) = z$, then we get $f(z) \cdot g(z) = 1$. Of course the function $f \cdot g$ has a removable singularity at the special point $z_0 = 0$. Furthermore, the function which results does not have a zero at the point 0. The general rule is: let z_0 be an isolated, not removable, singularity of the analytic function f. If there is some $n \in \mathbb{N}$ such that the function given by $f(z) \cdot (z - z_0)^n$ has a removable singularity at z_0, and the resulting function does not have a zero at z_0, then z_0 is a pole of order n.

Finally, an *essential* singularity is neither removable, nor is it a pole.

Definition 12. *Let $f : G \to \mathbb{C}$ be analytic, such that all it's isolated singularities are either removable or else they are poles. Then f is called a* meromorphic *function (defined in G).*

15 The Laurent Series

Since a pole of order n involves a function which looks somewhat like $(z - z_0)^{-n}$, at least near to the singularity z_0, it seems reasonable to expand our idea of power series into the negative direction. This gives us the Laurent series. That is, a sum which looks like this:

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

There is a little problem with this notation. After all, if the series is not absolutely convergent, then we might get different sums if we start at different places in the doubly infinite sequence. Let us therefore say that the sum from 0 to $+\infty$ is the *positive* series, and that from $-\infty$ to -1 is the *negative* series. So the whole series is absolutely convergent if both the positive and negative series are absolutely convergent.[16] One can obviously imagine that the negative series is really a positive series in the variable $1/(z-z_0)$.

So given some collection of coefficients c_n for all $n \in \mathbb{Z}$, let $R \geq 0$ be the radius of convergence of the series

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n$$

and let $1/r \geq 0$ be the radius of convergence of the series

$$\sum_{n=1}^{\infty} c_{-n} w^n = \sum_{n=-1}^{-\infty} c_n \left(\frac{1}{w}\right)^n .$$

That is to say, if $1/|z-z_0| < 1/r$, or put another way, if $|z-z_0| > r$, then the negative series

$$\sum_{n=-1}^{-\infty} c_n (z-z_0)^n$$

converges.

Therefore, there is an open ring (or *annulus*) of points of the complex plane, namely the region $\{z \in \mathbb{C} : r < |z-z_0| < R\}$, where the Laurent series is absolutely convergent. In this ring, the function defined by the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

is analytic. Note however that it is not necessarily true that f has an antiderivative. We see this by observing that the particular term $c_{-1}(z-z_0)^{-1}$ has no antiderivative in the ring. On the other hand, if we happen to have $c_{-1} = 0$ then there is an antiderivative, namely the function in the ring given by the Laurent series

$$\sum_{\substack{n=-\infty \\ n \neq -1}}^{\infty} \frac{c_n}{n+1} (z-z_0)^{n+1}.$$

Theorem 29. *Assume the Laurent series converges in the ring between* $0 \leq r < R \leq \infty$. *Let* $r < \rho < R$. *The coefficients of the Laurent series are then given by*

$$c_n = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

Proof. Since the series is uniformly convergent around the circle of radius ρ, we can exchange sum and integral signs to write

$$\int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{n+1}} dz = \sum_{k=-\infty}^{\infty} c_k \int_{|z-z_0|=\rho} \frac{(z-z_0)^k}{(z-z_0)^{n+1}} dz.$$

However, if we look at the individual terms, we see that if $k \neq n$, then each term has an antiderivative, and thus the path-integral is zero for that term. We are left with the n-term, and this is then simply

$$\int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{n+1}} dz = c_n \int_{|z-z_0|=\rho} \frac{1}{z-z_0} dz = 2\pi i \cdot c_n.$$

□

Conversely, we have:

[16] The negative series is called the "Hauptteil" in German, whilst the positive series is the "Nebenteil".

Theorem 30. *Given* $0 \le r < \rho \le R$, *let* $G = \{z \in \mathbb{C} : r < |z - z_0| < R\}$ *and* $f : G \to \mathbb{C}$ *be an analytic function. Then we have* $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, *where*

$$c_n = \frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{f(z)}{z^{n+1}} dz.$$

Proof. For simplicity, choose $z_0 = 0$. Let z be given with $r < |z| < R$, and take $\epsilon > 0$ such that $\epsilon < \min\{R - |z|, |z| - r\}$. Thus, according to theorems 5 and 6, we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{|\zeta - z|=\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta \\
&= \frac{1}{2\pi i}\int_{|\zeta|=R-\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i}\int_{|\zeta|=r+\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta \\
&= \frac{1}{2\pi i}\int_{|\zeta|=R-\epsilon} \frac{f(\zeta)}{\zeta}\left(\frac{1}{1-\frac{z}{\zeta}}\right) d\zeta + \frac{1}{2\pi i}\int_{|\zeta|=r+\epsilon} \frac{f(\zeta)}{z - \zeta} d\zeta \\
&= \frac{1}{2\pi i}\int_{|\zeta|=R-\epsilon} \frac{f(\zeta)}{\zeta}\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^n d\zeta + \frac{1}{2\pi i}\int_{|\zeta|=r+\epsilon} \frac{f(\zeta)}{z - \zeta} d\zeta \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|\zeta|=R-\epsilon} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta\right) z^n + \frac{1}{2\pi i}\int_{|\zeta|=r+\epsilon} \frac{f(\zeta)}{z - \zeta} d\zeta \\
&= \sum_{n=0}^{\infty} c_n z^n + \frac{1}{2\pi i}\int_{|\zeta|=r+\epsilon} \frac{f(\zeta)}{z - \zeta} d\zeta
\end{aligned}$$

So the terms for $n \ge 0$ are OK. Now let's look at the terms with $n < 0$. We have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{|\zeta|=r+\epsilon} \frac{f(\zeta)}{z - \zeta} d\zeta &= \frac{1}{2\pi i}\int_{|\zeta|=r+\epsilon} \frac{f(\zeta)}{z\left(1 - \frac{\zeta}{z}\right)} d\zeta \\
&= \frac{1}{z}\cdot\frac{1}{2\pi i}\int_{|\zeta|=r+\epsilon} f(\zeta)\sum_{n=0}^{\infty}\left(\frac{\zeta}{z}\right)^n d\zeta \\
&= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}\left(\frac{1}{2\pi i}\int_{|\zeta|=r+\epsilon} f(\zeta)\zeta^n d\zeta\right) \\
&= \sum_{n=-1}^{-\infty} c_n z^n.
\end{aligned}$$

(Note that theorem 5 shows that the integrals for $|z| = R - \epsilon$, and for $|z| = r + \epsilon$ are equal to the same integrals, taken along the path $|z| = \rho$.) □

Theorem 31. *Again, the same assumptions as in theorem 30. Assume further that there exists some* $M > 0$ *with* $|f(z)| \le M$ *for all* z *with* $|z - z_0| = \rho$. *Then* $|c_n| \le M/\rho^n$ *for all* n.

Proof.

$$|c_n| \le \frac{1}{2\pi}\int_{|z-z_0|=\rho}\left|\frac{f(z)}{z^{n+1}}\right| dz \le \frac{M}{\rho^n}.$$

□

Theorem 32 (Riemann). *Let* $a \in G \subset \mathbb{C}$ *be an isolated singularity of an analytic function* $f : G\backslash\{a\} \to \mathbb{C}$, *such that the exists an* $M > 0$ *and an* $\epsilon > 0$ *with* $|f(z)| \le M$ *for all* $z \in G$ *with* $z \ne a$ *and* $|z - a| < \epsilon$. *Then* a *is a removable singularity.*

Proof. For then $|c_n| \le M/r^n$ for all $0 < r < \epsilon$, and therefore, for the terms with $n < 0$ we must have $c_n = 0$, showing that in fact f is given by a normal power series, and thus it is also analytic in a. □

Theorem 33 (Casorati-Weierstrass). *Let a be an essential singularity of the function $f : G \setminus \{a\} \to \mathbb{C}$. Then for all $\epsilon, \delta > 0$ and $w \in \mathbb{C}$, there exists a $z \in G$ with $|z - a| < \epsilon$ such that $|f(z) - w| < \delta$. (Which is to say,* arbitrarily small *neighborhoods of a are "exploded" through the action of f throughout $\mathbb{C}$, so that they form a dense subset of $\mathbb{C}$!)*

Proof. Otherwise, there must exist some $w_0 \in \mathbb{C}$ such that there exists an $\epsilon > 0$ and $|f(z)-w_0| \geq \delta$ for all $z \in G$ with $|z-a| < \epsilon$. Let $B(a, \epsilon) = \{z \in \mathbb{C} : |z-a| < \epsilon\}$ and define the function $h : (B(a, \epsilon)\setminus\{a\}) \cap G \to \mathbb{C}$ to be

$$h(z) = \frac{1}{f(z) - w_0}.$$

Clearly h is analytic, with an isolated singularity at the point a. Furthermore,

$$|h((z)| \leq \frac{1}{\delta}.$$

Therefore, according to theorem 32 we must have a being removable. Thus, writing

$$f(z) = \frac{1}{h(z)} + w_0,$$

we see that since the function given by $1/h(z)$ has at most a pole at a, we cannot have a being an essential singularity of f. This is a contradiction. □

As an example of a function with an essential singularity, consider the function $f(z) = \exp(1/z)$. Clearly f is defined for all $z \neq 0$, and not defined for the single point 0. In fact, 0 is an essential singularity. To see this, consider the exponential series

$$f(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = 1 + \sum_{n=0}^{-\infty} \frac{z^n}{(-n)!}.$$

The singularity at 0 obviously cannot be a pole of the function, since the negative series is infinite. In fact we can make a theorem out of this observation.

Theorem 34. *Let a function f be defined by a Laurent series $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ around a point $z_0 \in \mathbb{C}$. Assume that the series converges in a "punctured disc" $\{0 < |z-z_0| < R\}$. If infinitely many of the terms c_n, for $n < 0$, are not zero, then z_0 is an essential singularity of f.*

Proof. Obviously z_0 is not a removable singularity. If it were a pole of order n, then all the terms c_m, for $m < -n$ must vanish. The only remaining possibility is that z_0 is an essential singularity. □

16 The Calculus of Residues

Let's begin by thinking about a function f with a pole of order m at the point $a \in \mathbb{C}$. That is, in a sufficiently small neighborhood of a we can write $h(z) = f(z)(z-a)^m$, and after filling in the removable singularity of h at a, we have $h(a) \neq 0$. Let

$$h(z) = \sum_{n=1}^{\infty} c_n(z - a)^n.$$

For $\epsilon > 0$ sufficiently small we therefore have

$$\begin{aligned}
\mathrm{Res}_a(f(z)) &= \frac{1}{2\pi i} \int_{|z-a|=\epsilon} f(z)dz \\
&= \frac{1}{2\pi i} \int_{|z-a|=\epsilon} \frac{h(z)}{(z-a)^m} dz \\
&= c_{m-1}
\end{aligned}$$

Or putting this another way, we can say that

$$\text{Res}_a(f(z)) = c_{m-1} = \frac{1}{(m-1)!}\frac{d^{(m-1)}}{dz^{(m-1)}}_{|z=a}\left((z-a)^m f(z)\right).$$

So this is a formula for the residue of a function with a pole at a.

A rather special case is the following. Let us assume that $G \subset \mathbb{C}$ is a region, and g, h are both analytic functions defined on G. Assume that $a \in G$ is a simple zero of h. (That is $h(a) = 0$, but $h'(a) \neq 0$.) Assume furthermore that $g(a) \neq 0$. Then let $f = g/h$. (Note that a function such as f, which is defined to be the ratio of two analytic functions, is called a *rational function*.) Therefore f is meromorphic in G. What is the residue of f at a? Writing g and h as power series around a, we have

$$g(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$$

and

$$h(z) = \sum_{l=1}^{\infty} d_l (z-a)^l.$$

Since a is a pole of order 1, we have

$$\begin{aligned}
\text{Res}_a(f(z)) &= \frac{1}{(1-1)!}\frac{d^0}{dz^0}\left((z-a)\frac{g(z)}{h(z)}\right)_{|z=a} \\
&= \left(\frac{(z-a)\sum_{k=0}^{\infty} c_k (z-a)^k}{\sum_{l=1}^{\infty} d_l (z-a)^l}\right)_{|z=a} \\
&= \left(\frac{\sum_{k=0}^{\infty} c_k (z-a)^k}{\sum_{l=1}^{\infty} d_l (z-a)^{l-1}}\right)_{|z=a} \\
&= \frac{c_0}{d_1} = \frac{g(a)}{h'(a)}.
\end{aligned}$$

Of course, going in the other direction, if a is a zero of g and $h(a) \neq 0$, then the residue of f at a is simply zero. This is trivial.

Theorem 35 (The Residue Theorem)**.** *Let the function* f *be defined and analytic throughout the region* $G \subset \mathbb{C}$*, except perhaps for a set* $S \subset G$ *of isolated, not removable singularities. Let* Ω *be a cycle in* G *which avoids all these singularities and which is such that the winding number of* Ω *around all points of the compliment of* G *is zero. Then only finitely many points of* S *have non-vanishing index with respect to* Ω *and we have the* residue formula

$$\frac{1}{2\pi i}\int_{\Omega} f(z)dz = \sum_{a \in S} \nu_{\Omega}(a)\text{Res}_a f(z).$$

Proof. Nothing is lost if we assume that Ω simply consists of a single closed path γ. So it is contained in a compact disc in $\mathbb{C}$ which must contain all points of $\mathbb{C}$ having a non-vanishing index with respect to γ. If there were infinitely many such points, then they must have an accumulation point, which must lie in the compliment of G. But such a point has index zero with respect to γ. Since that point does not lie on γ, it must have a neighborhood which contains only points with index zero with respect to γ. Thus we have a contradiction. The residue theorem now follows from Cauchy's theorem (theorem 26). □

17 Residues Around the Point at "Infinity"

In many applications, one speaks of the properties of a function "at infinity". For this, we take the "compactification" of the complex number plane. This is the set $\mathbb{C} \cup \{\infty\}$, where "$\infty$" is simply an abstract symbol. Then the open sets of $\mathbb{C} \cup \{\infty\}$ are, first of all the familiar open sets of $\mathbb{C}$, then in addition, we say that any set of the form $\{z \in \mathbb{C} : |z| > R\} \cup \{\infty\}$ is open, for $R > 0$. Finally, the union of all these sets of sets gives the topology of $\mathbb{C} \cup \{\infty\}$. It turns out that $\mathbb{C} \cup \{\infty\}$ is homeomorphic to the standard 2-sphere. It is often called the "Riemann sphere".

Definition 13. *Let* $G \subset \mathbb{C} \cup \{\infty\}$ *be a region containing* ∞. *(Thus* G *is open and connected.)*[17] *We will say that* $f : G \to \mathbb{C}$ *is analytic at* ∞ *if the function given by* $f(1/z)$ *has a removable singularity at* 0. *Similarly,* f *has a zero of order* n, *or a pole of order* n *at* ∞ *if the respective property is true of the function* $f(1/z)$ *at* 0. *That is to say, if* ∞ *is a zero of order* n *of* f *then we would like to have the function* $w^n f(w)$ *having a removable, non-zero singularity at* ∞. *However this is the same as looking at the limit as* $z \to 0$ *of the function which is given by substituting* $w = 1/z$. *That is, the function*

$$g(z) = \frac{1}{z^n} \cdot f\left(\frac{1}{z}\right).$$

So f *has a zero of order* n *at infinity if* g *has a removable, non-zero singularity at* 0.

Theorem 36. *Let* $g, h : \mathbb{C} \to \mathbb{C}$ *be entire, not constant, functions (thus they are analytic at all points of* $\mathbb{C}$*). Let* f *be the meromorphic function* $f = g/h$. *Assume that there is no zero of* h *on the real number line. Assume furthermore that* f *has a zero at infinity of order at least* 2. *Then we have*

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{\mathrm{Im}(a)>0} \mathrm{Res}_a f(z).$$

(Here the sum is over all poles of the function f *in the "upper" half plane of* $\mathbb{C}$. *That is, the set of all complex numbers with positive imaginary parts.)*

Proof. Because f has a zero at infinity, we have f being bounded outside of a compact disc of the form $D = \{z \in \mathbb{C} : |z| \geq r\}$, for some sufficiently large $r > 0$. In particular, all of the poles of f must be within D. Since the zeros of h are isolated, there are only finitely many of them, thus the sum over the poles is finite. Therefore, the residue theorem shows that

$$\int_\gamma f(z)dz = 2\pi i \sum_{\mathrm{Im}(a)>0} \mathrm{Res}_a f(z),$$

where γ is the closed curve which consists of two segments: first the segment along the real number line from $-r$ to r, then the segment consisting of the semi-circle of radius r around zero, traveling upwards from r and around through ir, then coming back to $-r$. That is to say,

$$2\pi i \sum_{\mathrm{Im}(a)>0} \mathrm{Res}_a f(z) = \int_{-r}^{r} f(x)dx + \int_{\alpha_r} f(z)dz,$$

where α_r is this semi-circle in the upper half-plane. But

$$\int_{\alpha_r} f(z)dz = \int_0^1 f(re^{\pi i t}) r\pi i e^{\pi i t} dt.$$

We are assuming that f has a zero at infinity of order at least 2. That means, for all $\epsilon > 0$ there exists some $\delta > 0$ such that for all $w \neq 0$ in $\mathbb{C}$ with $|w| < \delta$, we have

$$\left| \frac{1}{w} \cdot f\left(\frac{1}{w}\right) \right| < \epsilon.$$

Or put another way, $|z \cdot f(z)| < \epsilon$ for all $z \in \mathbb{C}$ with $|z| = r > 1/\delta$. Therefore

$$\left| \int_0^1 f(re^{\pi i t}) r\pi i e^{\pi i t} dt \right| < \int_0^1 \pi \cdot \epsilon dt = \pi \cdot \epsilon.$$

Since ϵ can be taken to be arbitrarily small, giving a corresponding δ, we can choose our r to be greater than $1/\delta$, and thus the integral around the half-circle α_r is small. The limit $r \to \infty$ gives the formula of the theorem. □

[17] Note that we must then have $\mathbb{C} \setminus G$ being a closed and bounded set, thus compact.

So this gives a method of calculating an integral along the real number line without actually having to do the integral at all! We only need to know the residues of the poles of the function in the upper half-plane of $\mathbb{C}$; no poles are allowed to be on the real number line; and the function should tend to zero sufficiently quickly at infinity.

But the assumption that the zero at infinity is of order 2 or more might be too restrictive. Perhaps the function we happen to be looking at only has a simple zero at infinity. For example consider the function

$$f(z) = \frac{1}{z-i}.$$

The integral along the real number line does not converge, and so this shows that we cannot expect the integral to exist if the zero at infinity is only of the first order. But perhaps the following theorem, where $f(x)$ is multiplied with the "rotating" function e^{ix}, thus mixing things up nicely, might be useful. However, in contrast to the case where the zero at infinity is of at least second order, here we cannot expect that the integral over the absolute value of the function also converges.

Theorem 37. *The same assumptions as in the previous theorem, except that the zero of* f *at infinity is only of the first order. Then we have*

$$\int_{-\infty}^{\infty} f(x)e^{ix}dx = 2\pi i \sum_{Im(a)>0} Res_a f(z)e^{iz}.$$

Proof. Again, we only have finitely many singularities in the upper half-plane. This time take a closed path γ consisting of four straight segments. The first segment is the straight line from the point $-r$ to $+r$, along the real number line. The second segment goes from r to $r+ir$. The third from $r+ir$ to $-r+ir$, and the fourth from $-r+ir$ back to the starting point at $-r$. Let's call these segments $\gamma_1(r), \dots, \gamma_4(r)$. Following the ideas in the proof of the previous theorem, we see that it is only necessary to show that

$$\lim_{r\to\infty} \int_{\gamma_j(r)} f(z)e^{iz}dz = 0,$$

for $j = 2, 3, 4$. Our assumption implies that for all $\epsilon > 0$ there exists an $r_0 > 0$ such that $|f(z)| < \epsilon$ for all z with $|z| > r_0$. We will show that the absolute value of the integral along the path $\gamma_2(r_0)$ is less than ϵ. The calculation for the other paths is similar. We have

$$\left| \int_{\gamma_2(r_0)} f(z)e^{iz}dz \right| \leq \int_0^{r_0} |f(r+it)|e^{-t}dt < \int_0^{r_0} \epsilon e^{-t}dt = \epsilon\left(1 - e^{-r_0}\right) < \epsilon.$$

□

18 Integrating Across a Pole

For example, consider the "integral"

$$\int_{-1}^{1} \frac{dx}{x}.$$

Obviously this is nonsense, since the integrals from -1 to 0, and from 0 to 1 of the function $1/x$ diverge. Specifically, for $0 < \epsilon < 1$ we have

$$\int_{\epsilon}^{1} \frac{dx}{x} = \log \epsilon.$$

By the same token, it is clear that

$$\int_{-1}^{-\epsilon} \frac{dx}{x} = -\log \epsilon.$$

Thus, if we agree to abandon the principles we have learned in the analysis lectures, and simply say that

$$\int_{-1}^{1} \frac{dx}{x} \stackrel{?}{=} \lim_{\epsilon\to 0} \left(\int_{-1}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{1} \frac{dx}{x} \right) = 0,$$

then we have the *Cauchy principle value* of the integral, and we see that in this case it is simply zero. One could make an important-looking definition here, but let us confine our attention to integrals along closed intervals $[a, b] \subset \mathbb{R}$ of complex-valued functions, where there might be poles of the function in the given interval. Assume for the moment there is a single pole at the point $p \in (a, b)$. Then we will define the *principle value* of the integral (if it exists) to be

$$\mathcal{P}\int_a^b f(x)dx \stackrel{\text{def}}{=} \lim_{\epsilon \searrow 0}\left(\int_a^{p-\epsilon} f(x)dx + \int_{p+\epsilon}^b f(x)dx\right).$$

Then the generalization to having a finite number of poles of f along the interval (but not at the endpoints) is clear.

Theorem 38. *Let* R *be a rational function, defined throughout* $\mathbb{C}$ *(together with it's poles). Assume that it has a zero at infinity, so that there can only be finitely many poles. Let* $p_1 < \cdots < p_m$ *be the poles of* R *which happen to lie on* $\mathbb{R}$*. Assume that each of these poles is simple; that is, of order* 1*. We distinguish two cases:*

- *If* R *has a simple zero at infinity (that is, of order* 1*), then we take* $f(z) = R(z)e^{iz}$*.*
- *Otherwise,* R *has a pole of order at least* 2 *at infinity, and in this case we take* $f(z) = R(z)$*.*

Then we have

$$\lim_{r\to\infty} \mathcal{P}\int_{-r}^{r} f(x)dx = 2\pi i \sum_{Im(a)>0} Res_a f(z) + \pi i \sum_{j=1}^{n} Res_{p_j} f(z).$$

Proof. In either case, we can have only finitely many poles of the function f; therefore only finitely many poles along the real number line. Let γ_δ be the path along the real number line from $-\infty$ to ∞, but altered slightly, following a semi-circle of radius δ above each of the poles on the real line. Furthermore, δ is sufficiently small that no other pole of f is enclosed within any of the semi-circles. Then, according to our previous theorems, we have

$$\int_{\gamma_\delta} f(x)dx = 2\pi i \sum_{Im(a)>0} Res_a f(z).$$

To simplify our thoughts, let us first consider the case that there is only one single pole $p \in \mathbb{R}$ on the real number line. And to simplify our thoughts even further, assume that $p = 0$. Then we have the path γ_δ coming from $-\infty$ to $-\delta$, then it follows the path $\delta e^{i\pi(1-t)}$, for t going from 0 to 1, and then finally it goes straight along the real number line from δ to ∞.

Let us now consider the Laurent series around 0. We can write

$$f(z) = \frac{c_{-1}}{z} + \sum_{n=0}^{\infty} c_n z^n = \frac{c_{-1}}{z} + g(z)$$

say. But then we can just define the new function g throughout $\mathbb{C}$ (leaving out the finite set of poles of f) by the rule

$$g(z) = f(z) - \frac{c_{-1}}{z}.$$

Obviously g has no pole at 0, but otherwise it has the same set of poles as the original function f. Since the function c_{-1}/z is analytic at all these other poles, the residue of g is identical with that of f around each of these poles. Therefore

$$\int_{-\infty}^{\infty} g(x)dx = \int_{\gamma_\delta} f(x)dx = 2\pi i \sum_{Im(a)>0} Res_a f(z).$$

On the other hand, since g is continuous at 0, we must have

$$\int_{-\infty}^{\infty} g(x)dx = \lim_{\delta\to 0}\int_{\gamma_\delta} g(x)dx.$$

Now, to get the principle value of the integral for f, we use the path γ_δ, but we must remove the semi-circle part of it. That is, let

$$\omega_\delta(t) = \delta e^{i\pi t},$$

for t from 0 to 1. Then

$$\mathcal{P}\int_{-\infty}^{\infty} f(x)dx = \lim_{\delta\to 0}\left(\int_{\gamma_\delta} f(x)dx + \int_{\omega_\delta} f(z)dz\right) = 2\pi i \sum_{Im(a)>0} Res_a f(z) + \lim_{\delta\to 0}\int_{\omega_\delta} f(z)dz.$$

But

$$\lim_{\delta\to 0}\int_{\omega_\delta} f(z)dz = \lim_{\delta\to 0}\int_{\omega_\delta}\left(\frac{c_{-1}}{z} + g(z)\right)dz = \lim_{\delta\to 0}\int_{\omega_\delta}\frac{c_{-1}}{z}dz = 2\pi i \cdot Res_0 f(z)\cdot \lim_{\delta\to 0}\int_{\omega_\delta}\frac{1}{z}dz.$$

This final integral is easy to calculate. We have

$$\int_{\omega_\delta}\frac{1}{z}dz = \int_0^1 \frac{i\pi\delta e^{i\pi t}}{\delta e^{i\pi t}} = i\pi.$$

Therefore the theorem is true in this case.

For the more general case, we take

$$g(z) = f(z) - \sum_{j=1}^{m}\frac{Res_{p_j} f(z)}{z - p_j},$$

and then proceed as before. □

19 Integrating Out From a Pole

Maybe we are dissatisfied with this "Cauchy principle value" technique. After all, it is rather like cheating! So let's see what we can do with an integral like

$$\int_0^\infty f(x)dx,$$

where 0 is a pole, and ∞ is a zero of f. For example, look at the function $f(x) = 1/x$. But here we see big problems! *Both* of the integrals $\int_1^\infty f(x)dx$ and $\int_0^1 f(x)dx$ are divergent.

Thinking about this, we see that the problems with the function $1/x$ stem from the fact that, first of all, the zero at ∞ is simple, and second of all, the pole at 0 is simple. This leads us to formulate the following theorem.

Theorem 39. *Again, let* R *be a rational function defined throughout* $\mathbb{C}$*, but this time with a zero of order at least* 2 *at* ∞*. Furthermore,* R *has no poles in the positive real numbers (*$x > 0$*), and at most a simple pole at* 0*. Then we have*

$$\int_0^\infty x^\lambda R(x)dx = \frac{2\pi i}{1 - e^{2\pi i\lambda}}\sum_{a\neq 0} Res_a z^\lambda R(z),$$

for all $0 < \lambda < 1$*.*

Proof. As before, the function $z^\lambda R(z)$ has at most finitely many poles in $\mathbb{C}$. Let $\gamma_{r,\phi}$, where $0 < r < 1$ and $0 < \phi < \pi$, be the following closed curve. It starts at the point $re^{\phi i}$ and follows a straight line out to the point $Te^{\phi i}$, where $T = 1/r$. Next it travels counter-clockwise around the circle of radius T, centered at 0, till it reaches the point $Te^{(2\pi-\phi)i}$. Next it travels along a straight line to the point $re^{(2\pi-\phi)i}$. Finally it travels back to the starting point, following the circle of radius r in a clockwise direction. Lets call these segments of the path L_1, L_2, L_3 and L_4. By choosing r and ϕ to be sufficiently small, we ensure that the path $\gamma_{r,\phi}$ encloses all poles of the function. In the exercises, we have seen that the path integrals along the segments L_2 and L_4 tend to zero for $r \to 0$. Thus we have

$$\lim_{r,\phi\to 0}\left(\int_{L_1} z^\lambda R(z)dz + \int_{L_3} z^\lambda R(z)dz\right) = 2\pi i\sum_{a\neq 0} Res_a z^\lambda R(z).$$

But we can take the path L_1 to be $te^{\phi i}$, for $t \in [r, T]$. Thus we have

$$\int_{L_1} z^\lambda R(z)dz = \int_r^T (te^{\phi i})^\lambda R(te^{\phi i})e^{\phi i}dt = e^{(\lambda+1)\phi i)} \int_r^T t^\lambda R(te^{\phi i})dt.$$

Similarly,

$$\int_{L_3} z^\lambda R(z)dz = \int_r^T (te^{(2\pi-\phi)i})^\lambda R(te^{(2\pi-\phi)i}) \left(-e^{(2\pi-\phi)i}\right) dt = -e^{(2\pi-\phi)(\lambda+1)i} \int_r^T t^\lambda R(te^{(2\pi-\phi)i})dt.$$

In the limit as $\phi \to 0$, we then have

$$\int_{L_1} z^\lambda R(z)dz + \int_{L_3} z^\lambda R(z)dz \to \left(1 - e^{2\pi i\lambda}\right) \int_r^T x^\lambda R(x)dx.$$

Finally, taking the limit as $r \to 0$ gives us the result. □

20 Symmetric Real Functions

For example, how do we calculate the integral

$$\int_0^\infty \frac{\sin x}{x} dx \ ?$$

On the one hand, there is no singularity at 0, but on the other hand, how are we to calculate the integral other than by using the calculus of residues?[18] Let us begin by noting that

$$\int_\epsilon^\infty \frac{\sin x}{x} dx = \int_\epsilon^\infty \frac{e^{ix} - e^{-ix}}{2ix} dx = \frac{1}{2i} \left(\int_{-\epsilon}^{-\infty} \frac{e^{ix}}{x} dx + \int_\epsilon^\infty \frac{e^{ix}}{x} dx \right) = \frac{1}{2i} \mathcal{P} \int_{-\infty}^\infty \frac{e^{ix}}{x} dx.$$

But the residue of $\exp(ix)/x$ at 0 is simply 1. Therefore, theorem 38 shows that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

This example shows that if we have a meromorphic function f which is defined in $\mathbb{C}$, such that $f(x) = f(-x)$ for all $x \in \mathbb{R}$, and such that the conditions of theorem 38 are satisfied, then it does make sense to calculate the integral $\int_0^\infty f(x)dx$, using the Cauchy principle value technique.

21 The Logarithmic Derivative

Let $f : G \to \mathbb{C}$ be a function. Thinking about the rules for derivatives, we can combine what we know about the derivative of a logarithm and the chain rule to arrive at the interesting observation that

$$(\log f(z))' = \frac{f'(z)}{f(z)}.$$

This is an interesting equation, particularly so in the field of analytical number theory. But the theorem we will look at here concerns path integrals. For simplicity, we will assume that $G = \mathbb{C}$, and that the function f is meromorphic, so that it is analytic everywhere in $\mathbb{C}$, except possibly for some set of isolated singularities. At each of these singularities, f has a pole of some order.

So let us say that f has a pole of order k at the point $a \in \mathbb{C}$. Then we can write

$$f(z) = \frac{g(z)}{(z-a)^k} = (z-a)^{-k} g(z),$$

[18] For example, partial integration only seems to make things more complicated here. Nevertheless, I see that the computer algebra system MuPAD does give the correct answer with little fuss!

where g is analytic at a, and $g(a) \neq 0$. Looking at the logarithmic derivative, we have

$$\frac{f'(z)}{f(z)} = \frac{-k(z-a)^{-k-1}g(z) + (z-a)^{-k}g'(z)}{(z-a)^{-k}g(z)} = \frac{-k}{z-a} + \frac{g'(z)}{g(z)}.$$

Therefore, if $r > 0$ is sufficiently small that the disc around a with radius r avoids all other poles and zeros of f, then we have

$$\frac{1}{2\pi i}\int_{|z-a|=r} \frac{f'(z)}{f(z)} dz = -k.$$

A similar calculation, where a is now a *zero* of order k of f, shows that

$$\frac{1}{2\pi i}\int_{|z-a|=r} \frac{f'(z)}{f(z)} dz = k.$$

Combining this with the residue theorem, we have a method of counting the zeros and poles. Namely:

Theorem 40. *Let f be a meromorphic function, defined in a region $G \subset \mathbb{C}$, and let Ω be some cycle which avoids all zeros and poles of f and which is such that all points of $\mathbb{C} \setminus G$ have index zero with respect to Ω. Let N_Ω be the number of zeros of f which have non-vanishing index with respect to Ω, counted according to their orders, and let P_Ω be the poles, again counted with their orders. Then*

$$\frac{1}{2\pi i}\int_{\Omega} \frac{f'(z)}{f(z)} dz = N_\Omega - P_\Omega.$$

Another way to look at this is to remember that f is, after all, just a mapping of a region of $\mathbb{C}$ back into $\mathbb{C}$. Thus if γ is a closed path in the region G (again, with winding number zero with respect to all points in the compliment of G in $\mathbb{C}$), it follows that $f \circ \gamma$, given by $f \circ \gamma(t) = f(\gamma(t))$, is itself another closed path in $\mathbb{C}$. If we assume that γ passes through no zero or pole of f, then the path $f \circ \gamma$ avoids both $0 \in \mathbb{C}$ and also the special point ∞. Therefore we can think about the index of 0 with respect to this path.

Theorem 41. $\nu_{f\circ\gamma}(0) = N_\gamma - P_\gamma$, *where the numbers N_γ and P_γ have been defined in the previous theorem (theorem 40).*

Proof. As we saw in theorem 25, we have

$$\nu_{f\circ\gamma}(0) = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{z} dz.$$

However

$$\frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{z} dz = \frac{1}{2\pi i}\int_a^b \frac{(f\circ\gamma)'(t)}{(f\circ\gamma)(t)} dt = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))} dt = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = N_\gamma - P_\gamma,$$

where we use the result of the previous theorem. Here, we have just imagined that the path γ is defined on some interval of the form $[a, b]$. Of course it is a trivial matter to see that we could replace the single path γ with a cycle Ω. □

Theorem 42 (Roché)**.** *Let $f, g : G \to \mathbb{C}$ be analytic functions, and let Ω be a cycle in G. Assume $A = \{w \in \mathbb{C} : \nu_\Omega(w) \neq 0\} \subset G$, and assume furthermore that $|g(z)| < |f(z)|$ for all z which lie directly on Ω. Then both functions f and $f + g$ have the same number of zeros in A.*

Proof. Let z be a point of Ω. Then since $0 \leq |g(z)| < |f(z)|$, we certainly do not have $f(z) = 0$. But also, for all $\tau \in [0, 1]$, we have

$$|f(z) + \tau g(z)| \geq |f(z)| - \tau|g(z)| > 0.$$

Therefore the cycle $(f + g) \circ \Omega$ is homotopic to the cycle $f \circ \Omega$ in $\mathbb{C} \setminus \{0\}$, and the result then follows from theorem 41. □

This theorem gives us a very quick proof of the Fundamental Theorem of Calculus. For let

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

be some polynomial of degree $n \geq 1$ in $\mathbb{C}$. Let $R = |a_{n-1}| + \cdots + |a_0| + 1$. Then on the circle given by $|z| = R$, we have

$$0 \leq |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| < |z^n|.$$

But then theorem 42 implies that $P(z)$ must have n zeros (counted with their multiplicities) within the circle of radius R.

22 Montel's Theorem

Thinking about Weierstrass' convergence theorem (theorem 28), let us again consider sequences of functions.

Definition 14. *Let $G \subset \mathbb{C}$ be a region, and for each $n \in \mathbb{N}$ let $f_n : G \to \mathbb{C}$ be analytic. This gives us a sequence of functions on G. We will say the sequence is* locally bounded *if for all $z \in G$, there exists an (open) neighborhood $z \in U \subset G$ and an $M > 0$, such that $|f_n(w)| \leq M$, for all $w \in U$ and all $n \in \mathbb{N}$.*

Theorem 43. *Let $f_n : G \to \mathbb{C}$ be a locally bounded sequence of analytic functions. Assume there exists a dense subset $T \subset G$, such that $(f_n(z))_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{C}$ for all $z \in T$. Then there exists an analytic function $f : G \to \mathbb{C}$ such that $f_n \to f$ uniformly on every compact subset of G.*

Proof. Begin by observing that we only need prove that for every $z_0 \in G$, there exists an $r > 0$ such that the sequence of functions f_n is uniformly convergent on $B(z_0, r)$ (the open disc of radius r centered on z_0). This follows, since given any compact subset $K \subset G$, it can be covered with finitely many such discs.

So given some $z_0 \in G$, we would like to show that there exists an $r > 0$ such that for all $\epsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that $|f_n(z) - f_m(z)| < \epsilon$ for all $n, m \geq N_0$ and $|z - z_0| < r$. Given this, then for each such z we would have $(f_n(z))_{n\in\mathbb{N}}$ being a Cauchy sequence, converging to a point $f(z)$ in $\mathbb{C}$. Thus the sequence would be uniformly convergent in $B(z_0, r)$ to the function f.

In order to find such an r and N_0, let us begin by using the property that the sequence of functions is locally bounded. Thus there is some $M > 0$ and an $r > 0$ such that $|f_n(z)| \leq M$ for all $z \in D(z_0, 2r) = \{z \in G : |z - z_0| \leq 2r\}$. Since T is dense in G, and $D(z_0, r)$ is compact, we can find some finite number of points of T in $B(z_0, r)$, call them $a_1, \ldots, a_k \in T$, with

$$B(z_0, r) \subset \bigcup_{l=1}^{k} B(a_l, \frac{\epsilon}{3} \cdot \frac{r}{2} \cdot \frac{1}{M}).$$

Let N_0 be sufficiently large that

$$|f_n(a_l) - f_m(a_l)| < \frac{\epsilon}{3}$$

for all $m, n \geq N_0$ and for all $l = 1, \ldots, k$. Choose any $z \in B(z_0, r)$. Then there exists some $l \in \{1, \ldots, k\}$ with $z \in B(a_l, \epsilon r/6M)$. Therefore, for $m, n \geq N_0$ we have

$$|f_n(z) - f_m(z)| \leq \underbrace{|f_n(z) - f_n(a_l)|}_{S_1} + \underbrace{|f_n(a_l) - f_m(a_l)|}_{S_2} + \underbrace{|f_m(a_l) - f_m(z)|}_{S_3}.$$

By assumption, we know that $S_2 < \epsilon/3$. Let's look at S_1 (clearly, S_3 is similar). We have

$$\begin{aligned} |f_n(z) - f_n(a_l)| &= \frac{1}{2\pi}\left|\int_{|\zeta - z_0| = 2r} \left(\frac{f_n(\zeta)}{\zeta - z} - \frac{f_n(\zeta)}{\zeta - a_l}\right) d\zeta\right| \\ &= \frac{1}{2\pi}|z - a_l|\left|\int_{|\zeta - z_0| = 2r} \frac{f_n(\zeta)}{(\zeta - z)(\zeta - a_l)} d\zeta\right| \\ &< \frac{1}{2\pi}|z - a_l|\frac{M}{r^2}2\pi(2r) \\ &= |z - a_l|\frac{2M}{r} < \frac{\epsilon}{3}. \end{aligned}$$

Note that the inequality in the third line follows because the radius of the circle is $2r$, and of course r is greater than both $|\zeta - z|$ and $|\zeta - a_l|$. Also the last inequality is due to the fact that we have assumed that $|z - a_l| < \epsilon r/6M$. □

Theorem 44 (Montel)**.** *Assume $f_n : G \to \mathbb{C}$ (with $n \in \mathbb{N}$) is a locally bounded sequence of analytic functions. Then there exists a subsequence which is uniformly convergent on every compact subset of G.*

Proof. Take some arbitrary sequence $\{a_1, a_2, \dots\}$ which is dense in G. Since the sequence of points $(f_n(a_1)_{n\in\mathbb{N}}$ is bounded, there exists a convergent subsequence, giving a subsequence $(f_{1n})_{n\in\mathbb{N}}$ of the sequence of functions. Next look at the sequence of points $(f_{1n}(a_1)_{n\in\mathbb{N}}$. Again, there is a convergent subsequence. And so forth. So for each $m \in \mathbb{N}$, we obtain a sequence of functions $(f_{mn})_{n\in\mathbb{N}}$ which is such that the sequence of points $(f_{mn}(a_l)_{n\in\mathbb{N}}$ converges, for all $l \le m$. Therefore the sequence of fuctions $(f_{nn})_{n\in\mathbb{N}}$ satisfies the conditions of theorem 43. □

We can use this theorem to find a criterion for the convergence of a sequence of functions as follows.

Theorem 45. *Again, let $f_n : G \to \mathbb{C}$ be a locally bounded sequence of analytic functions. Assume there exists some $z_0 \in G$ such that the sequences $(f_n^{(k)}(z_0))_{n\in\mathbb{N}}$ (that is, the sequences of k-th derivatives) converge, for all k. Then $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on all compact subsets of G.*

Proof. According to theorem 43, if $(f_n(z))_{n\in\mathbb{N}}$ is convergent for all $z \in G$, then the sequence of functions is uniformly convergent on all compact subsets of G, and we are finished. So let's assume that there exists some $a \in G$, such that the sequence of points $(f_n(a))_{n\in\mathbb{N}}$ is *not* convergent. But at least it must be bounded, so there must be two different subsequences of the sequence of functions, being convergent at a to two different values, say one subsequence converges to the value v_a and the other converges to w_a, where $w_a \neq v_a$. However, using Montel's theorem, we have subsequences of these subsequences of functions, converging to two *different* analytic functions: $f, g : G \to \mathbb{C}$, with $f(a) = w_a \neq v_a = g(a)$. Looking at the point z_0, we have

$$\lim_{n\to\infty} f_n^{(k)}(z_0) = f^{(k)}(z_0) = g^{(k)}(z_0),$$

that is, $(f - g)^k(z_0) = 0$ for all k. Therefore, the function $f - g$ is zero in a neighborhood of z_0, but this implies that it is zero everywhere, including at the point a, which gives us a contradiction. □

23 Infinite Products

After thinking about Weierstrass' theorem, where we are interested in infinite sums of analytic functions, the question comes up, is it also possible to deal with infinite products? Well it certainly is possible, and this is the subject of a number of classical theorems within complex analysis.

Before we get involved with infinite products of functions, we should first think about something easier, namely infinite products of numbers alone. So let $z_1, z_2, \dots$ be a sequence of numbers. These give rise to a sequence of "partial products"

$$P_n = \prod_{k=1}^{n} z_k.$$

But we should realize that there are some special things to think about here which make things different from the simpler situation with partial sums.

- For example with sums, the convergence of the series is not affected if we change a single term. But with products, if one of the terms z_k is changed to 0, then obviously all of the subsequent P_n are zero, regardless of what the further terms look like. Therefore we see that it only makes sense to consider products where *all* terms are non-zero.
- Another thing is that we could have $\lim_{n\to\infty} P_n = 0$. While this may not seem to be particularly objectionable at first, it is when one realizes that in this case, the limit again remains unchanged if various terms in the product are changed.

Both of these considerations show that, in a way, the number 0 in a product creates the same problems as does the number ∞ in a sum. So we will just agree to do away with the number zero when thinking about infinite products. However, because some people still find it nice to think about the number zero, the following definition will be used.

Definition 15. *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers which contains at most finitely many zeros. If the sequence of partial products of the non-zero terms converges to a number which is not zero, then we will say that the infinite product is convergent.*

It is a rather trivial observation that, for a convergent product of the form $\prod_{n\in\mathbb{N}} z_n$, we must have $\lim_{n\to\infty} z_n = 1$. Furthermore, we can assume that *at most* one of the z_n is a negative real number. For it is obvious that if we have two negative numbers, then it is simpler to just take the corresponding positive numbers. In fact, for this reason it is best to simply exclude negative real numbers from our considerations here completely, and if, as a very special case, we find it convenient to multiply things with the number -1, then that can be done at the end of our calculations.

This means that if we multiply numbers of the form z_n, then we will assume that we can write $z_n = r_n e^{i\theta_n}$, with $-\pi < \theta_n < \pi$. Or put another way, we can write $\log z_n = \log r + i\theta_n$. This is the *principal branch* of the logarithm.

Theorem 46. *Let $z_k = x_k + iy_k$ for all $k \in \mathbb{N}$ such that if $y_k = 0$ then $x_k > 0$. (That is, all complex numbers are allowed* except *for real numbers which are not positive.) Then we have that $\prod_{k=1}^{\infty} z_k$ is convergent if and only if $\sum_{k=1}^{\infty} \log z_n$ is convergent (where, of course, we take the principal branch of the logarithm).*

Proof. First assume that the sum of the logarithms converge. For example, let

$$\lim_{n\to\infty} \sum_{k=1}^{n} \log z_k = \alpha.$$

But then

$$\exp(\alpha) = \lim_{n\to\infty} \exp\left(\sum_{k=1}^{n} \log z_k\right) = \lim_{n\to\infty} \prod_{k=1}^{n} z_k,$$

so the product converges too. (Here we are simply using the fact that the exponential function is continuous.)

Going in the other direction, assume that the product of the z_k's converges. For example, let

$$\lim_{n\to\infty} \prod_{k=1}^{n} z_k = \beta \neq 0.$$

Writing the partial products as

$$P_n = \prod_{k=1}^{n} z_k,$$

we then have

$$\lim_{n\to\infty} \frac{P_n}{\beta} = 1.$$

Of course this implies that the sequence of fractions P_m/P_n converges to 1. Therefore let $N_0 \in \mathbb{N}$ be sufficiently large that

$$\left|\frac{P_m}{P_n} - 1\right| < \frac{1}{2},$$

for all m and $n \geq N_0$. In particular, for $m > n \geq N_0$, let $P_{n,m} = \prod_{k=n+1}^{m} z_k$. Then we have $|P_{n,m} - 1| < 1/2$. This means that $\mathrm{Re}(P_{n,m}) > 0$, that is, in particular, $P_{n,m} = re^{i\phi}$ say, with $-\pi/2 < \phi < +\pi/2$. Of course we always have $z_{m+1} = P_{m+1}/P_m$ so that $\mathrm{Re}(z_{m+1}) > 0$ for all $m > N_0$ as well. Thus, choosing $\log(z_k)$ to be in the principle branch of the logarithm for all $k \geq N_0$, we can write

$$\log\left(\prod_{k=N_0}^{m} z_k\right) = \sum_{k=N_0}^{m} \log z_k,$$

and we always remain in the principal branch of the logarithm. Since

$$\lim_{m\to\infty}\prod_{k=N_0}^{m} z_k = \frac{\beta}{\prod_{k=1}^{N_0-1} z_k}$$

exists, and it's logarithm has a unique value in the principal branch for all m, we must have

$$\sum_{k=N_0}^{\infty} \log z_k$$

also being convergent. Finally we can add on the finitely many terms from 1 to $N_0 - 1$. □

Of course, the logarithm always seems troublesome. Therefore the following theorem reduces things to a level which can be more easily checked.

Theorem 47. *Writing* $z_k = 1+a_k$, *we have* $\prod_{k=1}^{\infty}(1+a_k)$ *is absolutely convergent (that is, the sum of the absolute values of the logarithms,* $\sum_{k=1}^{\infty}|\log(1+a_k)|$ *is convergent) if and only if* $\sum_{k=1}^{\infty}|a_k|$ *converges.*

Proof. Begin by observing that since we have $\log'(z) = 1/z$, it follows that

$$\log'(1) = \lim_{a\to 0}\frac{\log(1+a)}{a} = 1.$$

Since $\log'$ is a continuous function, there exists a $\delta > 0$, such that for all $|a| < \delta$ we have

$$\frac{1}{2} < \left|\frac{\log(1+a)}{a}\right| < \frac{3}{2}.$$

That is,

$$\frac{|a|}{2} < |\log(1+a)| < \frac{3|a|}{2}.$$

This shows that $\sum_{k=1}^{\infty}|\log(1+a_k)|$ converges if and only if the sum $\sum_{k=1}^{\infty}|a_k|$ converges. □

24 Infinite Products of Functions

For example, we have seen in exercise 6.3 that for z not a real integer, we have the series

$$\sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2}$$

converging. In fact it is absolutely convergent. Therefore, according to theorem 47, we must have

$$\prod_{n\in\mathbb{Z}}\left(1+\frac{1}{(z-n)^2}\right)$$

converging, and so defining a function, for all $z \notin \mathbb{Z}$. But is this function meromorphic? It is a small exercise to see that the sequence of partial products is uniformly convergent on compact subsets of $\mathbb{C}\setminus\mathbb{Z}$, and thus according to Weierstrass' convergence theorem, the function is analytic in $\mathbb{C}\setminus\mathbb{Z}$. So this example shows one way to proceed in this special case.

On the other hand, polynomials seem to give us a more natural basis for generating analytic functions. Given a polynimial $P(z)$, we can write it as

$$P(z) = a(a_1 - z)\cdots(a_n - z),$$

where $a_1, \ldots, a_n$ are the zeros of the polynomial (perhaps some with multiplicity greater than one), and $a \neq 0$ is some constant. If we generalize this to an infinite product, and if we hope that things will converge, then we expect the terms to converge to 1. So it is natural to write something like this:

$$f(z) = \alpha \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right).$$

If the sequence of absolute values $|a_n|$ grows sufficiently rapidly, then we might expect to have convergence. But even a simple sequence like $a_n = n$ does not satisfy this property. Such considerations led to Weierstrass' product theorem.

Theorem 48 (Weierstrass). *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers with $a_n \neq 0$ for all n, and $\lim_{n\to\infty} a_n = \infty$. Then there exist integers $m_n \geq 0$ such that the product*

$$\Psi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}$$

converges to an entire function. The set of it's zeros is the sequence of the a_n (counted with their multiplicities).

Proof. Begin by observing that for $|w| < 1$, we have

$$\log(1 - w) = \sum_{k=1}^{\infty} -\frac{w^k}{k}.$$

Now we choose m_n, for each n, by specifying that m_n is sufficiently large that

$$\left| \log(1 - w) + \sum_{k=1}^{m_n} \frac{w^k}{k} \right| < \frac{1}{2^n},$$

for all w with $|w| < 1/2$. Since $\lim_{n\to\infty} a_n = \infty$, we have that for any $z \in \mathbb{C}$, there exists an $N_z \in \mathbb{N}$ such that $|a_n| > 2|z|$, for all $n \geq N_z$. Then we must have

$$g_z(v) = \sum_{n=N_z}^{\infty} \left(\log\left(1 - \frac{v}{a_n}\right) + \sum_{k=1}^{m_n} \frac{1}{k}\left(\frac{v}{a_n}\right)^k \right)$$

being absolutely and uniformly convergent for all v with $|v| \leq |z|$. Furthermore, as we have seen before, since $|v/a_n| < 1/2$, we stay in the principle branch of the logarithm in this sum.

By Weierstrass' convergence theorem, g_z is analytic for $|v| \leq |z|$, and $\exp\circ g_z$ is too. But what is $\exp\circ g_z(v)$? We have

$$\exp\circ g_z(v) = e^{g_z(v)} = \prod_{n=N_z}^{\infty} \left(1 - \frac{v}{a_n}\right) e^{\frac{v}{a_n} + \cdots + \frac{1}{m_n}\left(\frac{v}{a_n}\right)^{m_n}}.$$

This is never zero, since we always have $|a_n| > |v|$, for $n \geq N_z$. But now we multiply the remaining $N_z - 1$ terms onto this function, obtaining the analytic function

$$\Psi(v) = \left(\prod_{n=1}^{N_z - 1} \left(1 - \frac{v}{a_n}\right) e^{\frac{v}{a_n} + \cdots + \frac{1}{m_n}\left(\frac{v}{a_n}\right)^{m_n}} \right) \cdot \left(\prod_{n=N_z}^{\infty} \left(1 - \frac{v}{a_n}\right) e^{\frac{v}{a_n} + \cdots + \frac{1}{m_n}\left(\frac{v}{a_n}\right)^{m_n}} \right)$$

which, as we see, does not depend on the choice of z. □

In fact, we can be more specific than this.

Corollary. *Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Then there exists an entire function g such that*

$$f(z) = z^{m_0} e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}$$

for a_n the non-zero zeros of f and m_0 the order of the zero of f at 0 ($m_0 = 0$ if there is no zero at 0).

Proof. Writing

$$\Psi(z) = z^{m_0} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}},$$

we have that the function given by $\phi(z) = f(z)/\Psi(z)$ is an entire function (with removable singularities) with $\phi(z) \neq 0$, for all $z \in \mathbb{C}$. But then $\phi'(z)/\phi(z)$ is also an entire function, with antiderivative $g(z)$, say. Let's look at the derivative of the function $\phi(z)e^{-g(z)}$. We get

$$\left(\phi(z)e^{-g(z)}\right)' = \phi'(z)e^{-g(z)} - \phi(z)g'(z)e^{-g(z)} = \phi'(z)e^{-g(z)} - \phi(z)\frac{\phi'(z)}{\phi(z)}e^{-g(z)} = 0.$$

So $\phi(z)e^{-g(z)}$ is a constant, which we can absorb into g in such a way that $\phi(z)e^{-g(z)} = 1$ □

A simple consequence of this theorem is that every meromorphic function defined throughout $\mathbb{C}$ is a rational function. That is, let f be meromorphic, with poles $a_1, a_2, \ldots$.[19] Regardless of the way the poles are numbered, since they are isolated we must have $\lim_{n\to\infty} a_n = \infty$. Therefore take the product $\Psi \cdot f$, where Ψ is an entire function with the same set of zeros as f has poles (weighted according to their orders). Thus $\Psi \cdot f$ is — apart from a set of isolated, removable singularities — an entire function; call it $g : \mathbb{C} \to \mathbb{C}$. Therefore we can write $f = g/\Psi$.

Looking at the the formula in Weierstrass' theorem, we see that it can become a bit of a mess, particularly when the numbers m_n get larger and larger. Things are nicer if there exists a single number m, such that Weierstrass' theorem works with all $m_n \leq m$.

Definition 16. *Given a countable set of non-zero complex numbers $\{a_n\}$, if there exists an integer $m \geq 0$ such that the product*

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \cdots + \frac{1}{m}\left(\frac{z}{a_n}\right)^{m}}$$

converges, then this is called the canonical product *associated with the sequence $(a_n)_{n\in\mathbb{N}}$. The smallest such m is called the* genus *of the canonical product.*

When does the canonical product exist? That is to say, for a fixed non-negative integer m, and for each a_n, we have a "remainder term" of the form

$$r_n(z) = \log\left(1 - \frac{z}{a_n}\right) + \sum_{k=1}^{m} \frac{1}{k}\left(\frac{z}{a_n}\right)^k.$$

Looking at the proof of Weierstrass' theorem, we see that the canonical product (with respect to this m) will exist if

$$\sum_{n=N}^{\infty} r_n(z)$$

converges, for some sufficiently large N. For large enough n, we have $|z| < |a_n|$, and so

$$r_n(z) = -\sum_{k=m+1}^{\infty} \frac{1}{k}\left(\frac{z}{a_n}\right)^k.$$

Therefore

$$|r_n(z)| \leq \frac{1}{m+1}\left|\frac{z}{a_n}\right|^{m+1} \cdot \left(1 + \left|\frac{z}{a_n}\right| + \left|\frac{z}{a_n}\right|^2 + \cdots\right) = \frac{1}{m+1}\left|\frac{z}{a_n}\right|^{m+1} \cdot \left(\frac{1}{1 - |z/a_n|}\right).$$

So for a given z, and m fixed, we require that the series

$$\sum_{n=1}^{\infty} \frac{1}{m+1}\left|\frac{z}{a_n}\right|^{m+1}$$

[19] Of course if there are only finitely many poles, then this is trivial. On the other hand, since the poles are isolated, there can be at most countably many of them.

converges. This will be true if the series

$$\sum_{n=1}^{\infty} |a_n|^{-(m+1)}$$

converges.

25 Some Infinite Products

Genus zero: Given a sequence of non-zero points a_n with $\sum 1/|a_n|$ converging, we can simply say that

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)$$

gives us an example of a genus zero function with zeros just where we want them. Of course if we also want a zero at 0, then we can multiply this product with the factor Cz^l say, where l is the order of this zero, and C is a non-zero constant which we can choose as we like.

The sine function: Being more concrete, let us look at the function $\sin \pi z$. According to the definition of the sine function, we have

$$\sin \pi z = \frac{e^{\pi i z} - e^{-\pi i z}}{2i}.$$

But this can only be zero if $e^{i\pi z} = e^{-i\pi z}$. Writing $z = x + iy$, this means that $e^{-y} e^{i\pi x} = e^{y} e^{-i\pi x}$. In particular, $y = 0$ so that the zeros are just the familiar zeros which we know from real analysis, namely the integers, $\mathbb{Z}$. As we know, the series $\sum 1/n^2$ converges, thus the complex sine function must have genus one. Writing

$$\sin \pi z = z e^{g(z)} \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n},$$

the problem is then to determine the function g. For this, we take the logarithmic derivatives of both sides. We obtain

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right).$$

So now we must think about the cotangent function.

In the exercises we have seen that

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

Taking antiderivatives of both sides, we see that

$$(-\pi \cot \pi z)' = \frac{\pi^2}{\sin^2 \pi z}$$

and

$$\left(\frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) \right)' = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

(Note here that

$$\frac{1}{z-n} + \frac{1}{n} = \frac{z}{n(z-n)}$$

so that the sum is uniformly convergent in compact subsets of $\mathbb{C}$ which do not contain points of $\mathbb{Z}$, thus showing that the derivative of the sum is the sum of the derivatives.)

Therefore

$$-\pi\cot\pi z = \underbrace{\frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right)}_{\chi(z)} + C,$$

where C is a constant. On the other hand, we have both $\cot(w) = -\cot(-w)$ and $\chi(w) = -\chi(-w)$. That is, they are both anti-symmetric. But this can only be true if the constant $C = 0$. But this implies that our function g satisfies the equation $g'(z) = 0$; that is, g is a constant. Since

$$\lim_{z\to 0}\frac{\sin\pi z}{z} = \pi,$$

we must have $e^{g(z)} = \pi$. Therefore, we end up with the representation

$$\sin\pi z = \pi z\prod_{n\neq 0}\left(1 - \frac{z}{n}\right)e^{z/n}.$$

26 The Gamma Function: I

In the analysis lecture, we defined the gamma function for real numbers $x > 1$ using the improper integral

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt.$$

We can do the same thing here in the realms of complex analysis. Just substitute the complex number z for the real number x in the formula. Using the same argument as in the real case, we find that the integral converges if $\mathrm{Re}(z) > 1$. But by doing this, we miss out on the fact that the gamma function can be defined everywhere in $\mathbb{C}$ (with isolated poles). Of course it is again possible to look at the functional equation for the gamma function, then bring in analytic continuation. But this is a rather complicated manipulation! In reality, an infinite product representation is simpler.

Let's begin by looking at the following function

$$G(z) = \prod_{n=-1}^{-\infty}\left(1 - \frac{z}{n}\right)e^{\frac{z}{n}} = \prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)e^{-\frac{z}{n}}.$$

We will be using the second expression here, although the first expression shows that we can apply theorem 48 and conclude that the product does converge to an entire function, whose zeros are simply the set of negative real integers. Taking a look at the product representation of the sine function which we obtained in the last section, we see that

$$zG(z)G(-z) = \frac{\sin\pi z}{\pi}.$$

The function $zG(z)$ obviously has as it's zeros the negative real integers, and also 0. But then, if we take the function $G(z-1)$, we see that it also has the negative real integers and 0 as it's set of zeros. Since all of these zeros are simple zeros for both functions, we have

$$\frac{G(z-1)}{zG(z)}$$

being an entire function which is never zero. It is now an exercise to show that there must exist an entire function $\gamma : \mathbb{C} \to \mathbb{C}$ such that

$$ze^{\gamma(z)}G(z) = G(z-1)$$

for all $z \in \mathbb{C}$.

Let's take the logarithmic derivative on both sides of the equation. We get

$$\begin{aligned}
\frac{1}{z}+\gamma'(z)+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right) &= \sum_{n=1}^{\infty}\left(\frac{1}{z-1+n}-\frac{1}{n}\right)\\
&= \frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n+1}\right)\\
&= \frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\
&= \frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)+1\\
&= \frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right).
\end{aligned}$$

Note that here

$$\frac{1}{n}-\frac{1}{n+1}=\frac{1}{n(n+1)}<\frac{1}{n^2},$$

so the sum is certainly absolutely convergent and thus the third equation is valid. But this means that $\gamma'(z)=0$, that is, $\gamma(z)$ is a constant, which we simply denote by γ. In fact it is *Euler's constant.*

To see this, consider the case $z=1$. From the formula defining G, we certainly have $G(0)=1$. Therefore $G(0)=1=e^{\gamma}G(1)$, or[20]

$$e^{-\gamma}=G(1)=\prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)e^{-\frac{1}{k}}=\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)e^{-\frac{1}{k}}=\lim_{n\to\infty}(n+1)e^{-(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})}.$$

Therefore

$$\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right).$$

So G is an entire function satisfying the functional equation $G(z-1)=ze^{\gamma}G(z)$. In order to improve the appearance of things, let us define $H(z)=G(z)e^{z\cdot\gamma}$. Then we have the functional equation

$$H(z-1)=zH(z).$$

This is beginning to look like the functional equation for the gamma function, but unfortunately (or fortunately?) it is going in the "wrong" direction. To fix this up, we take

$$\Gamma(z)=\frac{1}{zH(z)},$$

giving us *Euler's gamma function*

$$\Gamma(z+1)=z\Gamma(z).$$

Thus

$$\Gamma(z)=\frac{e^{-z\cdot\gamma}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{\frac{z}{n}}.$$

[20]Note here that we have

$$\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)=n+1.$$

This is easily proved using induction on n. The inductive step is to observe that

$$(n+1)\left(1+\frac{1}{n+1}\right)=n+1+\frac{n}{n+1}+\frac{1}{n+1}=(n+1)+1.$$

From the construction, we see that it has simple poles at the negative integers, and at 0, but it has no zeros. Also, looking at the equation for the sine function, we can express this in terms of the gamma function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

In particular, we have

$$\Gamma(1/2) = \sqrt{\pi}.$$

27 The Gamma Function: II

But what is the relationship with the formula

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt,$$

for x a real number greater than 1, which we used in the analysis lectures? Going from real to complex numbers, let us write $z = x + iy$. Then we have

$$\left|t^{z-1}\right| = \left|e^{((x-1)+iy)\log(t)}\right| = \left|e^{(x-1)\log(t)} \cdot e^{iy\log(t)}\right| = \left|e^{(x-1)\log(t)}\right| = \left|t^{x-1}\right|.$$

Therefore — as we saw in Analysis I — the integral defining the gamma function will again converge when $x > 1$; for $x < 1$ it diverges.

As an exercise, we see that for $z = x + iy$ and $x > 0$, we have

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt = \frac{n^z n!}{z(z+1)\cdots(z+n)},$$

for all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n = e^{-t},$$

we therefore[21] get

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)\cdots(z+n)}.$$

This is another standard formula for the gamma function, and in fact it is valid as long as $\mathrm{Re}(z) > 0$. But we still haven't shown that it is the same gamma function as that which was defined in the previous section.

For each $n \in \mathbb{N}$, let

$$g_n(z) = \frac{z(z+1)\cdots(z+n)}{n! n^z}.$$

This is obviously an entire function. On the other hand, at least for $\mathrm{Re}(z) > 0$, we have

$$\Gamma(z) = \lim_{n\to\infty} \frac{1}{g_n(z)}.$$

So the question is, what is this limit? Begin by making things look somewhat more complicated:

$$g_n(z) = z e^{z\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log(n)\right)} \prod_{k=1}^{n} \left(\left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}\right).$$

---

[21] For $0 \le t \le n$ we have

$$\left(1 - \frac{t}{n}\right)^n \le e^{-t}.$$

This can be seen by taking the logarithm. We have

$$\log\left(1 - \frac{t}{n}\right)^n = -\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{t}{n}\right)^k \le -t = \log\left(e^{-t}\right).$$

Therefore

$$\lim_{n\to\infty} g_n(z) = ze^{z\cdot\gamma}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}} = ze^{z\cdot\gamma}G(z) = zH(z),$$

and we see the connection with the definition in the previous section.

As a final remark before proceeding with Stirling's formula, I should mention that Legendre introduced the *beta function*, which is

$$\beta(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

This seems to be used these days in computer algebra programs. Also, when on that subject, one finds many interesting and obscure formulas related to the gamma function. (You can also look in books containing tables of mathematical formulas.) One such formula which caught my eye is the following. Let $n \in \mathbb{N}$. Then

$$\Gamma\left(\frac{n}{2}\right) = \frac{(n-2)!!\sqrt{\pi}}{2^{(n-1)/2}}.$$

Here $n!!$ is the *double factorial*. That is

$$n!! = \begin{cases} n(n-2)\cdots 5\cdot 3\cdot 1, & n>0 \text{ and odd},\\ n(n-2)\cdots 6\cdot 4\cdot 2, & n>0 \text{ and even},\\ 1, & n = 0,-1.\end{cases}$$

28 Stirling's Formula

The functional equation for the gamma function, $\Gamma(z+1) = z\Gamma(z)$, together with the observation that $\Gamma(1) = 1$, shows that for all $n \in \mathbb{N}$, we have $\Gamma(n+1) = n!$. In the analysis lecture, I gave a very simple approximation to the gamma function. Namely, for large integers n, we have $n! \approx n^n e^{-n}$. But this is really a very inexact approximation, which we were able to derive with hardly any thought at all. In this section, we will be looking at the "genuine" Stirling formula. It is

$$\Gamma(z) \sim z^{z-1/2}e^{-z}\sqrt{2\pi}.$$

More specifically, we prove that

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi - \int_0^{\infty}\frac{P_1(t)}{z+t}dt,$$

where $P_1(t) = t - [t] - 1/2$, and $[t]$ is the largest integer which is less than or equal to t. Of course we take the principle value of the logarithm here. The fact that, as long as z is not a negative real number, the integral on the right-hand side of the equation goes to zero as $|z| \to \infty$, shows that

$$\lim_{|z|\to\infty}\frac{z^{z-1/2}e^{-z}\sqrt{2\pi}}{\Gamma(z)} = 1$$

(at least if we stay away from the negative real numbers). In fact we will show that for a given fixed δ between 0 and π, we have that $P_1(z)$ falls uniformly to 0, for $z = re^{i\theta}$ and $|\theta| \le \pi - \delta$.

Lemma 1 (Euler's Summation formula)**.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be continuously differentiable (in the sense of real analysis). Then*

$$\sum_{k=0}^{n} f(k) = \int_0^n f(t)dt + \frac{1}{2}(f(n)+f(0)) + \int_0^n P_1(t)f'(t)dt.$$

Proof. For k an integer, using partial integration and the observation that $P_1'(t) = 1$ when $t \notin \mathbb{Z}$, we have

$$\int_{k-1}^{k} P_1(t)f'(t)dt = \frac{P_1(k)f(k) + P_1(k-1)f(k-1)}{2} - \int_{k-1}^{k} f(t)dt.$$

Taking the sum from $k = 1$ to n then gives the result. □

Similarly we have

Lemma 2. *Assume* z *is not a negative real number or zero. Let* $P_2(t) = t_*(t_* - 1)/2$, *where* $t_* = t - [t]$ *for* $t \in \mathbb{R}$. *Then we have*

$$\Phi(z) = \int_0^\infty \frac{P_1(t)}{z+t}dt = \int_0^\infty \frac{P_2(t)}{(z+t)^2}dt.$$

The function Φ *which is so defined is analytic in* $\mathbb{C}$ *(minus the negative reals and* 0*).*

Proof. Using partial integration[22], we have

$$\int_{k-1}^{k} \frac{P_1(t)}{z+t}dt = \int_{k-1}^{k} \frac{P_2(t)}{(z+t)^2}dt,$$

for each $k \in \mathbb{N}$.Let us take $z = x + iy$, with $y \neq 0$. Then we have

$$\begin{aligned}
\left|\int_0^\infty \frac{P_1(t)}{z+t}dt\right| &\leq \int_0^\infty \left|\frac{P_2(t)}{(z+t)^2}\right| dt \\
&\leq \int_0^\infty \left|\frac{1}{(z+t)^2}\right| dt \\
&= \int_0^\infty \frac{dt}{y^2 + (t+x)^2} \\
&= \underbrace{\left|\int_0^{-x} \frac{dt}{y^2 + (t+x)^2}\right|}_{K} + \int_{-x}^\infty \frac{dt}{y^2 + (t+x)^2} \\
&= K + \int_0^\infty \frac{dt}{y^2 + t^2} = K + \frac{1}{y^2}\int_0^\infty \frac{dt}{1 + \left(\frac{t}{y}\right)^2} = K + \frac{1}{y}\int_0^\infty \frac{dt}{1+t^2} = K + \frac{\pi}{2y}
\end{aligned}$$

where K is a finite number. (When z is a positive real number, the integral obviously also converges.) Thus we can use Weierstrass' convergence theorem — on the integrals between successive integers — to conclude that Φ is an analytic function. □

Lemma 3. *We have*

$$\lim_{y \to \infty} \int_0^\infty \frac{P_1(t)}{iy + t}dt = 0.$$

Proof. This follows from the previous lemma, since then $K = 0$ and $\pi/2y \to 0$, when $y \to \infty$. □

So now that we have looked at these lemmas, let us begin proving Stirling's formula. It involves the standard, rather awkward problem of the logarithms. So to avoid this, let us first take the case that z is just a positive real number. That is, we take $z = x + i \cdot 0$, and we set about examining $\log \Gamma(x)$. Obviously $\Gamma(x) > 0$, so we can use the simple, real logarithm function. The antiderivative of $\log(x)$ is

[22] Here we are using the quotient rule to obtain the following derivative:

$$\left(\frac{t(t-1)}{2(z+t)}\right)' = \frac{2t-1}{2(z+t)} - \frac{2t(t-1)}{4(z+t)^2}.$$

$x\log(x) - x$. Therefore, using lemma 1, we see that for $n \in \mathbb{N}$ we have

$$\begin{aligned}
\log \frac{x(x+1)\cdots(x+n)}{n!n^x} &= \sum_{k=0}^{n} \log(x+k) - \sum_{k=0}^{n} \log(k+1) + \log(n+1) - x\log(n) \\
&= \int_0^n \log(x+t)dt - \int_0^n \log(t+1)dt + \frac{1}{2}(\log(x+n) + \log(x) - \log(n+1)) \\
&\quad + \log(n+1) - x\log(n) + \int_0^n \frac{P_1(t)}{x+t}dt - \int_0^n \frac{P_1(t)}{t+1}dt \\
&= (x+n)\log(x+n) - (x+n) - x\log(x) + x - (n+1)\log(n+1) + (n+1) - 1 \\
&\quad + \log(n+1) - x\log(n) + \frac{1}{2}(\log(x+n) + \log(x) - \log(n+1)) \\
&\quad + \int_0^n \frac{P_1(t)}{x+t}dt - \int_0^n \frac{P_1(t)}{t+1}dt
\end{aligned}$$

At this stage, we note that

$$x\log(x+n) = x\log n\left(1+\frac{x}{n}\right) = x\log n + x\log\left(1+\frac{x}{n}\right).$$

Furthermore, for $n > x$ we have

$$\log\left(1+\frac{x}{n}\right) = \frac{x}{n} - \frac{1}{2}\left(\frac{x}{n}\right)^2 + \cdots.$$

Therefore, we get relations such as

$$\lim_{n\to\infty} n\log\left(1+\frac{x}{n}\right) = x.$$

After an unpleasant calculation (which I will not reproduce here in TEX), using the expression for $\Gamma(x)$ which was found in the previous section, we end up with the equation

$$\log\Gamma(x) = \left(x - \frac{1}{2}\right)\log x - x - \Phi(x) + 1 + \int_0^n \frac{P_1(t)}{t+1}dt.$$

Lemma 2 shows that the last expression,

$$1 + \int_0^n \frac{P_1(t)}{t+1}dt,$$

is simply some constant real number, let's call it C. But then we see that the left-hand side of the equation, namely $\log\Gamma(x)$ is just the restriction to the positive real numbers of an analytic function which is defined in the region consisting of $\mathbb{C}$ with the negative real numbers and 0 removed. Similarly the right-hand side, namely $x\log x - (\log x)/2 - x - \Phi(x) + C$ is the restriction to the positive real numbers of another analytic function which is defined in the same region. Since both of these functions coincide on the positive real numbers, they must be same analytic function.

So the only remaining problem is to find out what the value of C is. Writing

$$\Gamma(1-z) = -z\Gamma(-z),$$

we have

$$\Gamma(z)\Gamma(-z) = -\frac{\Gamma(z)\Gamma(1-z)}{z} = -\frac{\pi}{z\sin\pi z}.$$

Choosing in particular $z = 0 + i\cdot y$, we observe[23] that $|\Gamma(iy)| = |\Gamma(-iy)|$. Therefore

$$|\Gamma(iy)| = \sqrt{\frac{2\pi}{y\left(e^{\pi y} - e^{-\pi y}\right)}}.$$

[23]For example, we can use the expression

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)\cdots(z+n)}.$$

Since $C \in \mathbb{R}$ we must have

$$C = \mathrm{Re}\left(\log\Gamma(iy) - \left(iy - \frac{1}{2}\right)\log(iy) + iy + \Phi(iy)\right).$$

This is true for all $y > 0$, no matter how large. But according to lemma 3, we have $\lim_{y\to\infty}\Phi(iy) = 0$. For the rest, we should remember that for a complex number of the form $re^{i\theta}$, we have

$$\mathrm{Re}\left(\log(re^{i\theta})\right) = \mathrm{Re}(\log r + i\theta) = \log r.$$

Therefore

$$C = \lim_{y\to\infty}\left(\log(|\Gamma(iy)|) + \frac{1}{2}\log(y) + \frac{\pi y}{2}\right) = \lim_{y\to\infty}\log\sqrt{\frac{2\pi y e^{\pi y}}{y\left(e^{\pi y} - e^{-\pi y}\right)}} = \frac{1}{2}\log(2\pi).$$

29 The Order of an Entire Function

Theorem 49. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be an entire function. Assume that there exist real constants* $C > 0$, $\lambda > 0$, *such that* $\mathrm{Re}(f(z)) \le C(1 + |z|^\lambda)$, *for all* $z \in \mathbb{C}$. *Then* f *is a polynomial, at most of degree* $[\lambda]$.

Proof. To begin with, we know that, for $k, l \in \mathbb{Z}$, we have

$$\int_0^{2\pi}\cos(k\theta)\sin(l\theta) = 0$$

and

$$\int_0^{2\pi}\cos(k\theta)\cos(l\theta) = \int_0^{2\pi}\sin(k\theta)\sin(l\theta) = \begin{cases} 0, & k \ne l \\ \pi, & k = l. \end{cases}$$

If $f(0) \ne 0$, then we can just substitute the function $f - f(0)$ for f. Thus we can assume without loss of generality that $f(0) = 0$. Developing f in a power series around 0, we write

$$f(z) = \sum_{n=1}^{\infty}(a_n + ib_n)z^n$$

where a_n and b_n are real numbers. Therefore

$$\mathrm{Re}(f(z)) = \sum_{n=1}^{\infty} r^n(a_n\cos(n\theta) - b_n\sin(n\theta)),$$

where $z = re^{i\theta}$. So for each $k \in \mathbb{N}$ we have

$$\int_0^{2\pi}\cos(k\theta)\mathrm{Re}(f(z))d\theta = a_k r^k\pi.$$

Similarly

$$\int_0^{2\pi}\sin(k\theta)\mathrm{Re}(f(z))d\theta = b_k r^k\pi.$$

But also (recalling theorem 7)

$$\int_0^{2\pi}\mathrm{Re}(f(z))d\theta = f(0) = 0.$$

Therefore we get

$$\begin{aligned}
|a_k| &\le \frac{1}{\pi r^k}\int_0^{2\pi}|\mathrm{Re}(f(z))|d\theta \\
&= \frac{1}{\pi r^k}\int_0^{2\pi}(|\mathrm{Re}(f(z))| + \mathrm{Re}(f(z)))d\theta \\
&= \frac{2}{\pi r^k}\int_0^{2\pi}\max(\mathrm{Re}(f(z)), 0)d\theta \\
&\le \frac{4C(1 + r^\lambda)}{r^k}
\end{aligned}$$

Therefore, taking $r \to \infty$, we see that if $k > \lambda$ then $a_k = 0$. An analogous argument shows also that $b_k = 0$. □

Definition 17. *An entire function* f *is said to have* finite order *if there exists some real number* $\rho > 0$, *and a constant* $C > 0$, *such that*

$$|f(z)| \leq Ce^{|z|^\rho},$$

for all $z \in \mathbb{C}$. *The infimum over all such* ρ *is the* order *of* f. *That is to say,* α *is the order of* f *if* $|f(z)| \leq Ce^{|z|^{\alpha+\epsilon}}$ *for all* $\epsilon > 0$ *and* $z \in \mathbb{C}$. *If* $|f(z)| \leq Ce^{|z|^\alpha}$ *for all* $z \in \mathbb{C}$ *then* α *is the* strict order *of* f.

Theorem 50. *Let* f *be an entire function of finite order with no zeros. Then* $f = e^g$, *where* g *is a polynomial whose degree is the order of* f.

Proof. According to exercise 12.1, given f, then there exists an entire function g with $f = e^g$. But then we must have

$$|f(z)| = \left|e^{g(z)}\right| = \left|e^{\mathrm{Re}(g(z))+i\mathrm{Im}(g(z))}\right| = \left|e^{\mathrm{Re}(g(z))}\right|\left|e^{i\mathrm{Im}(g(z))}\right| = \left|e^{\mathrm{Re}(g(z))}\right|.$$

So $\mathrm{Re}(g(z)) \leq |z|^\alpha$ if α is the order of f, and therefore the result follows from theorem 49. □

Going beyond this, we would like to think about entire functions of finite order, but *with* zeros. This leads us to Hadamard's theorem. But before we arrive there, let us think about Jensen's formula.

30 Jensen's Formula

But before doing that, we look at the easier *Jensen's inequality.*

Theorem 51 (Jensen's Inequality). *Let* $R > 0$ *be given and let the (non-constant) analytic function* f *be defined in a region containing the closed disc* $D_R = \{z \in \mathbb{C} : |z| \leq R\}$. *Assume* $f(0) \neq 0$ *and also* $f(z) \neq 0$ *for all* z *with* $|z| = R$. *Let the zeros of* f *in* D_R *be* $z_1, \ldots, z_n$. *(Here a zero is listed* m *times if it is a zero of order* m.*) We assume the zeros are ordered according to their increasing absolute value. Let* $\|f\|_R = \max\{|f(z)| : |z| = R\}$. *Then we have*

$$|f(0)| \leq \frac{\|f\|_R}{R^n}|z_1 \cdots z_n|.$$

More generally, thinking about various values of R, *let* $\nu_f(R) = n$ *be the number of zeros of* f *in* D_R, *where* R, *thus* n, *is allowed to vary. Then we have Jensen' inequality:*

$$\int_0^R \frac{\nu_f(t)}{t}dt \leq \log\|f\|_R - \log|f(0)|.$$

Proof. Consider the function

$$g(z) = f(z)\prod_{k=1}^n \frac{R^2 - z\bar{z}_k}{R(z - z_k)}.$$

It is obviously analytic in D_R. Furthermore, we have $|g(z)| = |f(z)|$ when $|z| = R$. This implies[24] that $|g(w)| \leq \|f\|_R$ for all $w \in D_R$. Therefore

$$|g(0)| = \left|f(0)\prod_{k=1}^n \frac{R}{z_k}\right| \leq \|f\|_R.$$

Taking logarithms of these real numbers, we have

$$\log\frac{R^n}{|z_1 \cdots z_n|} = \sum_{k=1}^n(\log R - \log|z_k|) = \sum_{k=1}^n \int_{|z_k|}^R \frac{dt}{t} = \int_0^R \frac{\nu_f(t)}{t}dt \leq \log\|f\|_R - \log|f(0)|.$$

□

[24]If we had a point w_* in the interior of D_R with $|g(w_*)| > \|f\|_R$, then we can assume that it is maximal with respect to this property. However, that would contradict theorem 19.

Theorem 52 (Jensen's Formula)**.** *The same assumptions as in the previous theorem. Then we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta}|d\theta = \log|f(0)| + \sum_{k=1}^{n}\log\frac{R}{|z_k|}.$$

Proof. In this proof, we will first look at two very simple cases:

1. We first prove Jensen's formula in the simple case that there are no zeros of f in D_R. Then again, as in exercise 12.1, we have an analytic function g, defined in a neighborhood of D_R, with $f = e^g$. Or put another way, $g = \log f$. (To be definite, we could specify that $\log f(0)$ should be in the principle branch of the logarithm.) Then Cauchy's formula is simply

   $$\log f(0) = \frac{1}{2\pi i}\int_{|z|=R}\frac{\log f(z)}{z}dz = \frac{1}{2\pi}\int_0^{2\pi}\log f(Re^{i\theta})d\theta.$$

   Taking the real part, we get

   $$|\log f(0)| = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}(\log f(Re^{i\theta}))d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta}))|d\theta,$$

   which establishes the theorem in this first case.

2. The second case is even simpler. Namely, let ζ be a complex number with $|\zeta| < R$. That is, ζ is some point in the interior of D_R. This second case is that the function f is simply $f(z) = z - \zeta$. We then define a new function, namely

   $$Q(z) = \frac{f(z)}{R^2 - z\bar{\zeta}} = \frac{z-\zeta}{R^2 - z\bar{\zeta}}.$$

   For $|z| = R$, we have $z\bar{z} = R^2$, so that then

   $$|Q(z)| = \left|\frac{z-\zeta}{z(\bar{z}-\bar{\zeta})}\right| = \frac{1}{R}.$$

   Thus

   $$\frac{1}{2\pi}\int_0^{2\pi}\log\left|Q(Re^{i\theta})\right|d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log\left(\frac{1}{R}\right)d\theta = -\log R.$$

   On the other hand, remembering that $f(z) = z - \zeta$, we have[25]

   $$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log\left|Q(Re^{i\theta})\right|d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(Re^{i\theta})\right|d\theta - \frac{1}{2\pi}\int_0^{2\pi}\log\left|R^2 - z\bar{\zeta}\right|d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(Re^{i\theta})\right|d\theta - \log R^2\\ &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(Re^{i\theta})\right|d\theta - 2\log R.\end{aligned}$$

   But this means[26] that we have

   $$\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(Re^{i\theta})\right|d\theta = \log R = \log\frac{R|\zeta|}{|\zeta|} = \log\left(|f(0)|\cdot\frac{R}{|\zeta|}\right) = \log|f(0)| + \log\frac{R}{|\zeta|}.$$

   So the theorem is established in the case $f(z) = z - \zeta$.

The more general case is that there are the zeros $z_1, \ldots, z_n$ in D_R. In this case, we have

$$f(z) = (z - z_1)\cdots(z - z_n)F(z),$$

where F has no zeros in D_R. Taking the logarithm of the product gives a sum, thus establishing the theorem in the general case as well. □

[25]Note that for the second equation here, we are using the result of case 1. of our proof. Obviously the function (of z) given by $R^2 - z\bar{\zeta}$ has no zeros in B_R.

[26]Remember that $|f(0)| = |0 - \zeta| = |\zeta|$.

31 Functions of Finite Order With Zeros

Theorem 53. *Let* f *be an entire function of strict order* α*. Then there exists a* $C > 0$ *such that, for all* $R > 0$ *and* $D_R = \{z \in \mathbb{C} : |z| \le r\}$*, we have*

$$\nu_f(R) \le CR^\alpha,$$

where $\nu_f(R)$ *is the number of zeros of* f *in* D_R*.*

Proof. Assume first that $f(0) \neq 0$. According to Jensen's inequality (leaving out the part where we integrate from 0, and remembering that $|f(z)| \le Ke^{|z|^\alpha}$, for some $K > 0$), we have, for $R > 1$

$$\int_R^{2R} \frac{\nu_f(t)}{t} dt \le \log \|f\|_{2R} \le \log Ke^{(2R)^\alpha} = 2^\alpha R^\alpha \log K.$$

However,

$$\nu_f(R) \log 2 = \nu_f(R) \int_R^{2R} \frac{dt}{t} \le \int_R^{2R} \frac{\nu_f(t)}{t} dt.$$

Finally, if we do have $f(0) = 0$, then take $g(z) = f(z)/z^m$, where m is the order of the zero at 0. Then the theorem will apply to g, and since m is then fixed, to f as well. □

Theorem 54. *Again,* f *is an entire function of strict order* α *with zeros* $\{z_n\}$*, listed (with multiplicity) in order of increasing absolute value. We assume that* $f(0) \neq 0$*. Then for every* $\delta > 0$ *we have the series*

$$\sum |z_n|^{-\alpha-\delta}$$

converging.

Proof. Using partial summation[27] and the previous theorem, where we assume that $R \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{|z_n| \le R} |z_n|^{\alpha+\delta} &\le \sum_{k=1}^{R} \frac{\nu_f(k+1) - \nu_f(k)}{k^{\alpha+\delta}} \\
&= \frac{\nu_f(R+1)}{(R+1)^{\alpha+\delta}} - \frac{\nu_f(1)}{1^{\alpha+\delta}} - \sum_{k=1}^{R} \nu_f(k)\left((k+1)^{\alpha+\delta} - k^{\alpha+\delta}\right) \\
&= \frac{\nu_f(R+1)}{(R+1)^{\alpha+\delta}} - \nu_f(1) + \sum_{k=1}^{R} \frac{1}{\alpha+\delta} \int_k^{k+1} \frac{\nu_f(k)}{t^{\alpha+\delta+1}} dt \\
&\le \frac{\nu_f(R+1)}{(R+1)^{\alpha+\delta}} - \nu_f(1) + \frac{1}{\alpha+\delta} \sum_{k=1}^{R} \frac{\nu_f(k)}{k^{\alpha+\delta+1}} \\
&\le CR^{-\delta} - \nu_f(1) + \frac{C}{\alpha+\delta} \sum_{k=1}^{R} \frac{1}{k^{1+\delta}},
\end{aligned}$$

and this last sum is convergent. □

This, combined with the discussion concerning the genus of a canonical product of discrete elements of $\mathbb{C}$ (see section 24), leads to:

Corollary (Hadamard)**.** *Let* f *be an entire function of finite order* α *with zeros* $\{z_n\}_{n \in \mathbb{N}}$*. Then*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \cdots + \frac{1}{m_*}\left(\frac{z}{z_n}\right)^{m_*}},$$

where m *is the order of the zero at* 0*,* g *is a polynomial of degree at most* α*, and* $m_* > \alpha - 1$*.*

[27] That is, given two sequences, $a_m, \ldots, a_{n+1}$ and $b_m, \ldots, b_{n+1}$, we have

$$\sum_{k=m}^{n} a_k(b_{k+1} - b_k) = [a_{n+1}b_{n+1} - a_m b_m] - \sum_{k=m}^{n} b_{k+1}(a_{k+1} - a_k).$$