11.18 Write a computer program for the finite element method with linear splines and test it for various examples. Compare the numerical results with those for the finite difference method.

11.19 Let $B_{-1}, B_0, B_1, \ldots, B_n, B_{n+1}, B_{n+2}$ denote the cubic B-splines for the equidistant grid $x_j := a + jh$, $j = 0, \ldots, n+1$, with step size $h = (b-a)/n$. Show that

$$u_0 := B_0 - 4B_{-1}, \ u_1 := B_1 - B_{-1},$$

$$u_2 := B_2, \ldots, u_{n-1} := B_{n-1},$$

$$u_n := B_n - B_{n+2}, \ u_{n+1} := B_n - 4B_{n+2}$$

is a basis for $S_3^n \cap H_0^1[a,b]$, i.e., for the space of cubic splines vanishing at the endpoints.

Using this basis, set up the Galerkin equations for the Sturm–Liouville problem analogous to the case of linear splines treated in Section 11.5.

11.20 Formulate and prove analogues of Theorems 11.27 and 11.28 for the finite element approximation using cubic splines as in Problem 11.19.

12
Integral Equations

from: R. Kress,
"Numerical Analysis"
(Springer, 1998).

The topic of the last chapter of this book is linear integral equations, of which

$$\int_a^b K(x,y)\varphi(y)\,dy = f(x), \quad x \in [a,b],$$

and

$$\varphi(x) - \int_a^b K(x,y)\varphi(y)\,dy = f(x), \quad x \in [a,b],$$

are typical examples. In these equations the function φ is the unknown, and the so-called *kernel* K and the right-hand side f are given functions. The above equations are called *Fredholm integral equations* of the *first* and *second kind*, respectively. Since both the theory and the numerical approximations for integral equations of the first kind are far more complicated than for integral equations of the second kind, we will confine our presentation to the latter case.

Integral equations provide an important tool for solving boundary value problems for both ordinary and partial differential equations (see Problem 12.1 and [39]). Their historical development is closely related to the solution of boundary value problems in potential theory in the last decades of the nineteenth century. Progress in the theory of integral equations also had a great impact on the development of functional analysis.

Omitting the proofs, we will present the main results of the Riesz theory for compact operators as the foundation of the existence theory for integral equations of the second kind. Then we will develop the fundamental ideas of the Nyström method and the collocation method as the two most important approaches for the numerical solution of these integral equations. This is done in a general framework of operator equations and their approximate solution, which makes the analysis more widely applicable. For a comprehensive study of both the theory and the numerical solution of linear integral equations we refer to [39].

12.1 The Riesz Theory

This section is devoted to a summary of some of the basic facts of the theory of Fredholm integral equations of the second kind. The integral equations formulated above carry the name of Fredholm, since in 1902 Fredholm established an existence theory for integral equations of the second kind with continuous kernels, which is now known as the Fredholm alternative. For the purpose of this introduction to the numerical solution of integral equations it suffices to consider only the first and most important part of this alternative, which states that the inhomogeneous equation

$$\varphi(x) - \int_a^b K(x,y)\varphi(y)\,dy = f(x), \quad x \in [a,b], \tag{12.1}$$

with continuous kernel K has a unique solution $\varphi \in C[a,b]$ for each right-hand side $f \in C[a,b]$ if and only if the homogeneous integral equation

$$\varphi(x) - \int_a^b K(x,y)\varphi(y)\,dy = 0, \quad x \in [a,b], \tag{12.2}$$

has only the trivial solution. The importance of this result originates from the fact that it reduces the difficult problem of establishing existence of a solution to the inhomogeneous integral equation to the simpler problem of showing that the homogeneous integral equation allows only the trivial solution $\varphi = 0$, and it extends the corresponding statement for systems of linear equations to the case of integral equations. Actually, Fredholm derived his results by interpreting integral equations as a limiting case of linear systems by considering the integral as a limit of Riemann sums and passing to the limit in Cramer's rule for the solution of linear systems. For the solution of integral equations with continuous kernels, Fredholm's approach is still the most elegant and shortest. However, since it is restricted to the case of continuous kernels, it is more convenient to consider the above equations as a special case of operator equations of the second kind with a compact operator, as presented by Riesz in 1918.

Definition 12.1 *A linear operator $A : X \to Y$ from a normed space X into a normed space Y is called* compact *if for each bounded sequence (φ_n) in X the sequence $(A\varphi_n)$ contains a convergent subsequence in Y, i.e., if each sequence from the set $\{A\varphi : \varphi \in X,\ \|\varphi\| \le 1\}$ contains a convergent subsequence.*

Without developing the concept of compactness in normed spaces in any detail, we note that this definition is equivalent to requiring that the set $\{A\varphi : \varphi \in X,\ \|\varphi\| \le 1\}$ be *relatively sequentially compact.*

Compact operators are bounded, linear combinations of compact operators are compact, and products of two bounded operators are compact if one of them is compact (see Problem 12.2). From the Bolzano–Weierstrass theorem it can be seen that bounded operators $A : X \to X$ with finite-dimensional range $A(X) := \{A\varphi : \varphi \in X\}$ are compact. Furthermore, the identity operator $I : X \to X$, defined by $I : \varphi \mapsto \varphi$ for all $\varphi \in X$, is compact if and only if the space X is finite-dimensional. This actually justifies the distinction between the equations $A\varphi = f$ and $\varphi - A\varphi = f$ as equations of the first and second kind, since A and $I - A$ have different properties in infinite-dimensional spaces if A is compact. A proof of these facts and of the following important theorem can be found in most introductory books on functional analysis, for example in [39].

The fundamental result of the Riesz theory is described by the following theorem, which extends Fredholm's result on the equivalence of injectivity and surjectivity to the case of operator equations of the second kind with a compact operator.

Theorem 12.2 *Let $A : X \to X$ be a compact operator in a normed space X. Then $I - A$ is surjective if and only if it is injective. If the inverse operator $(I - A)^{-1} : X \to X$ exists, it is bounded.*

In order to verify that Fredholm's existence analysis for integral equations with continuous kernels $K : [a,b] \times [a,b] \to \mathbb{R}$ can be viewed as a special case of Theorem 12.2, we have to establish that the linear integral operator $A : C[a,b] \to C[a,b]$, defined by

$$(A\varphi)(x) := \int_a^b K(x,y)\varphi(y)\,dy, \quad x \in [a,b], \tag{12.3}$$

is compact. For this we need the following theorem due to Arzelà–Ascoli, which again is proven in most introductions to functional analysis.

Theorem 12.3 (Arzelà–Ascoli) *Each sequence from a subset $U \subset C[a,b]$ contains a uniformly convergent subsequence; i.e., U is relatively sequentially compact, if and only if it is bounded and* equicontinuous, *i.e., if there exists a constant C such that*

$$|\varphi(x)| \le C$$

for all $x \in [a,b]$ and all $\varphi \in U$, and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\varphi(x) - \varphi(y)| < \varepsilon$$

for all $x, y \in [a,b]$ with $|x - y| < \delta$ and all $\varphi \in U$.

Theorem 12.4 *The integral operator (12.3) with continuous kernel is a compact operator on $C[a,b]$.*

Proof. For all $\varphi \in C[a,b]$ with $\|\varphi\|_\infty \le 1$ and all $x \in [a,b]$, we have that

$$|(A\varphi)(x)| \le (b-a) \max_{x,y\in[a,b]} |K(x,y)|;$$

i.e., the set $U := \{A\varphi : \varphi \in C[a,b],\ \|\varphi\|_\infty \le 1\} \subset C[a,b]$ is bounded. Since K is uniformly continuous on the square $[a,b]\times[a,b]$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|K(x,z) - K(y,z)| < \frac{\varepsilon}{b-a}$$

for all $x, y, z \in [a,b]$ with $|x-y| < \delta$. Then

$$|(A\varphi)(x) - (A\varphi)(y)| = \left| \int_a^b [K(x,z) - K(y,z)]\varphi(z)\,dz \right| < \varepsilon$$

for all $x, y \in [a,b]$ with $|x-y| < \delta$ and all $\varphi \in C[a,b]$ with $\|\varphi\|_\infty \le 1$; i.e., U is equicontinuous. Hence A is compact by the Arzelà–Ascoli Theorem 12.3. □

In our analysis we also will need an explicit expression for the norm of the integral operator A.

Theorem 12.5 *The norm of the integral operator $A : C[a,b] \to C[a,b]$ with continuous kernel K is given by*

$$\|A\|_\infty = \max_{a\le x\le b} \int_a^b |K(x,y)|\,dy. \tag{12.4}$$

Proof. For each $\varphi \in C[a,b]$ with $\|\varphi\|_\infty \le 1$ we have

$$|(A\varphi)(x)| \le \int_a^b |K(x,y)|\,dy, \quad x \in [a,b],$$

and thus

$$\|A\|_\infty = \sup_{\|\varphi\|_\infty \le 1} \|A\varphi\|_\infty \le \max_{a\le x\le b} \int_a^b |K(x,y)|\,dy.$$

Since K is continuous, there exists $x_0 \in [a,b]$ such that

$$\int_a^b |K(x_0,y)|\,dy = \max_{a\le x\le b} \int_a^b |K(x,y)|\,dy.$$

For $\varepsilon > 0$ choose $\psi \in C[a,b]$ by setting

$$\psi(y) := \frac{K(x_0,y)}{|K(x_0,y)| + \varepsilon}, \quad y \in [a,b].$$

ψ approaches $\operatorname{sgn}(K(x_0,y))$

Then $\|\psi\|_\infty \le 1$ and

$$\|A\psi\|_\infty \ge |(A\psi)(x_0)| = \int_a^b \frac{[K(x_0,y)]^2}{|K(x_0,y)|+\varepsilon}\,dy \ge \int_a^b \frac{[K(x_0,y)]^2 - \varepsilon^2}{|K(x_0,y)|+\varepsilon}\,dy$$

$$= \int_a^b |K(x_0,y)|\,dy - \varepsilon(b-a).$$

Hence

$$\|A\|_\infty = \sup_{\|\varphi\|_\infty \le 1} \|A\varphi\|_\infty \ge \|A\psi\|_\infty \ge \int_a^b |K(x_0,y)|\,dy - \varepsilon(b-a),$$

and since this holds for all $\varepsilon > 0$, we have

$$\|A\|_\infty \ge \int_a^b |K(x_0,y)|\,dy = \max_{a\le x\le b} \int_a^b |K(x,y)|\,dy.$$

This concludes the proof. □

It also can be shown that the integral operator remains compact if the kernel K is merely weakly singular (see [39]). A kernel K is said to be *weakly singular* if it is defined and continuous for all $x, y \in [a,b]$, $x \ne y$, and there exist positive constants M and $\alpha \in (0,1]$ such that

$$|K(x,y)| \le M|x-y|^{\alpha-1}$$

for all $x, y \in [a,b]$, $x \ne y$.

12.2 Operator Approximations

The fundamental concept for approximately solving an operator equation

$$\varphi - A\varphi = f$$

of the second kind is to replace it by an equation

$$\varphi_n - A_n\varphi_n = f_n$$

with approximating sequences $A_n \to A$ and $f_n \to f$ as $n \to \infty$. For computational purposes, the approximating equations will be chosen such that they can be reduced to solving a system of linear equations. In this section we will provide a convergence and error analysis for such approximation schemes. In particular, we will derive convergence results and error estimates for the cases where we have either norm or pointwise convergence of the sequence $A_n \to A$, $n \to \infty$.

A_n norm-convergent

Theorem 12.6 *Let $A : X \to X$ be a compact linear operator on a Banach space X such that $I - A$ is injective. Assume that the sequence $A_n : X \to X$ of bounded linear operators is norm convergent, i.e., $\|A_n - A\| \to 0$, $n \to \infty$. Then for sufficiently large n the inverse operators $(I - A_n)^{-1} : X \to X$ exist and are uniformly bounded. For the solutions of the equations*

$$\varphi - A\varphi = f \quad \textit{and} \quad \varphi_n - A_n\varphi_n = f_n$$

we have an error estimate

$$\|\varphi_n - \varphi\| \le C\{\|(A_n - A)\varphi\| + \|f_n - f\|\} \tag{12.5}$$

for some constant C.

Proof. By the Riesz Theorem 12.2, the inverse $(I - A)^{-1} : X \to X$ exists and is bounded. Since $\|A_n - A\| \to 0$, $n \to \infty$, by Remark 3.25 we have $\|(I - A)^{-1}(A_n - A)\| \le q < 1$ for sufficiently large n. For these n, by the Neumann series Theorem 3.48, the inverse operators of

$$I - (I - A)^{-1}(A_n - A) = (I - A)^{-1}(I - A_n)$$

exist and are uniformly bounded by

$$\|[I - (I - A)^{-1}(A_n - A)]^{-1}\| \le \frac{1}{1 - q}.$$

But then $[I - (I - A)^{-1}(A - A_n)]^{-1}(I - A)^{-1}$ are the inverse operators of $I - A_n$ and they are uniformly bounded.

The error estimate follows from

$$(I - A_n)(\varphi_n - \varphi) = (A - A_n)\varphi + f_n - f$$

by the uniform boundedness of the inverse operators $(I - A_n)^{-1}$. □

A_n ptwise convergent

In order to develop a similar analysis for the case where the sequence (A_n) is merely pointwise convergent, i.e., $A_n\varphi \to \varphi$, $n \to \infty$, for all φ, we will have to bridge the gap between norm and pointwise convergence. This goal will be achieved through the concept of collectively compact operator sequences and the following *uniform boundedness principle.*

(next 3 pages)

Theorem 12.7 *Let the sequence $A_n : X \to Y$ of bounded linear operators mapping a Banach space X into a normed space Y be pointwise bounded; i.e., for each $\varphi \in X$ there exists a positive number C_φ depending on φ such that $\|A_n\varphi\| \le C_\varphi$ for all $n \in \mathbb{N}$. Then the sequence (A_n) is uniformly bounded; i.e., there exists some constant C such that $\|A_n\| \le C$ for all $n \in \mathbb{N}$.*

Proof. In the first step, by an indirect proof we establish that positive constants M and ρ and an element $\psi \in X$ can be chosen such that

$$\|A_n\varphi\| \le M \tag{12.6}$$

for all $\varphi \in X$ with $\|\varphi - \psi\| \le \rho$ and all $n \in \mathbb{N}$. Assume that this is not possible. Then, by induction, we construct sequences (n_k) in $\mathbb{N}$, (ρ_k) in $\mathbb{R}$, and (φ_k) in X such that

$$\|A_{n_k}\varphi\| \ge k$$

for $k = 0, 1, 2, \dots$ and φ with $\|\varphi - \varphi_k\| \le \rho_k$ and

$$0 < \rho_k \le \frac{1}{2}\,\rho_{k-1}, \quad \|\varphi_k - \varphi_{k-1}\| \le \frac{1}{2}\,\rho_{k-1}$$

for $k = 1, 2, \dots$.

We initiate the induction by setting $n_0 = 1$, $\rho_0 = 1$, and $\varphi_0 = 0$. Assume that $n_k \in \mathbb{N}$, $\rho_k > 0$, and $\varphi_k \in X$ are given. Then there exist $n_{k+1} \in \mathbb{N}$ and $\varphi_{k+1} \in X$ satisfying $\|\varphi_{k+1} - \varphi_k\| \le \rho_k/2$ and $\|A_{n_{k+1}}\varphi_{k+1}\| \ge k + 2$. Otherwise, we would have $\|A_n\varphi\| \le k+2$ for all $\varphi \in X$ with $\|\varphi - \varphi_k\| \le \rho_k/2$ and all $n \in \mathbb{N}$, and this contradicts our assumption. Set

$$\rho_{k+1} := \min\left(\frac{\rho_k}{2}, \frac{1}{\|A_{n_{k+1}}\|}\right) \le \frac{\rho_k}{2}.$$

Then for all $\varphi \in X$ with $\|\varphi - \varphi_{k+1}\| \le \rho_{k+1}$, by the triangle inequality we have

$$\|A_{n_{k+1}}\varphi\| \ge \|A_{n_{k+1}}\varphi_{k+1}\| - \|A_{n_{k+1}}(\varphi - \varphi_{k+1})\| \ge k + 1,$$

since $\|A_{n_{k+1}}(\varphi - \varphi_{k+1})\| \le \|A_{n_{k+1}}\|\rho_{k+1} \le 1$.

For $j > k$, using the geometric series we have

$$\|\varphi_k - \varphi_j\| \le \|\varphi_k - \varphi_{k+1}\| + \dots + \|\varphi_{j-1} - \varphi_j\|$$

$$\le \frac{1}{2}\,\rho_k + \dots + \frac{1}{2}\,\rho_{j-1} \le \rho_k.$$

Therefore, (φ_k) is a Cauchy sequence and converges to some element φ in the Banach space X. From $\|\varphi_k - \varphi_j\| \le \rho_k$ for all $j \ge k$ by passing to the limit $j \to \infty$ we see that $\|\varphi_k - \varphi\| \le \rho_k$ for all $k \in \mathbb{N}$. Therefore, we have $\|A_{n_k}\varphi\| \ge k$ for all $k \in \mathbb{N}$, which is a contradiction to the boundedness of the sequence $(A_n\varphi)$.

Now, in the second step, from the validity of (12.6) we deduce for each $\varphi \in X$ with $\|\varphi\| \le 1$ and for all $n \in \mathbb{N}$ the estimate

$$\|A_n\varphi\| = \frac{1}{\rho}\,\|A_n(\rho\varphi + \psi) - A_n\psi\| \le \frac{2M}{\rho}\,.$$

This completes the proof. □

Following Anselone [2], we introduce the concept of collectively compact operator sequences.

Definition 12.8 *A sequence $A_n : X \to Y$ of linear operators from a normed space X into a normed space Y is called* collectively compact *if each sequence from the set $\{A_n\varphi : \varphi \in X, \|\varphi\| \le 1, n \in \mathbb{N}\}$ contains a convergent subsequence.*

note: both φ and n vary, together.

Each operator A_n from a collectively compact sequence is compact.

Lemma 12.9 *Let X be a Banach space, let $A_n : X \to X$ be a collectively compact sequence, and let $B_n : X \to X$ be a pointwise convergent sequence with limit operator $B : X \to X$. Then*

$$\|(B_n - B)A_n\| \to 0, \quad n \to \infty. \tag{12.7}$$

Proof. Assume that (12.7) is not valid. Then there exist $\varepsilon_0 > 0$, a sequence (n_k) in $\mathbb{N}$ with $n_k \to \infty$, $k \to \infty$, and a sequence (φ_k) in X with $\|\varphi_k\| \le 1$ such that

$$\|(B_{n_k} - B)A_{n_k}\varphi_k\| \ge \varepsilon_0, \quad k = 1, 2, \ldots. \tag{12.8}$$

Since the sequence (A_n) is collectively compact, there exists a subsequence such that

$$A_{n_{k(j)}}\varphi_{k(j)} \to \psi \in X, \quad j \to \infty. \tag{12.9}$$

Then we can estimate with the aid of the triangle inequality and Remark 3.25 to obtain

$$\begin{aligned} &\|(B_{n_{k(j)}} - B)A_{n_{k(j)}}\varphi_{k(j)}\| \\ &\le \|(B_{n_{k(j)}} - B)\psi\| + \|B_{n_{k(j)}} - B\|\,\|A_{n_{k(j)}}\varphi_{k(j)} - \psi\|. \end{aligned} \tag{12.10}$$

The first term on the right-hand side of (12.10) tends to zero as $j \to \infty$, since the operator sequence (B_n) is pointwise convergent. The second term tends to zero as $j \to \infty$, since the operator sequence (B_n) is uniformly bounded by Theorem 12.7 and since we have the convergence (12.9). Therefore, passing to the limit $j \to \infty$ in (12.10) yields a contradiction to (12.8), and the proof is complete. □

Theorem 12.10 *Let $A : X \to X$ be a compact linear operator on a Banach space X such that $I - A$ is injective, and assume that the sequence $A_n : X \to X$ of linear operators is collectively compact and pointwise convergent; i.e., $A_n\varphi \to A\varphi$, $n \to \infty$, for all $\varphi \in X$. Then for sufficiently large n the inverse operators $(I - A_n)^{-1} : X \to X$ exist and are uniformly bounded. For the solutions of the equations*

$$\varphi - A\varphi = f \quad \textit{and} \quad \varphi_n - A_n\varphi_n = f_n$$

we have an error estimate

$$\|\varphi_n - \varphi\| \le C\{\|(A_n - A)\varphi\| + \|f_n - f\|\} \tag{12.11}$$

for some constant C.

identical to (12 in norm-conv

Proof. By the Riesz Theorem 12.2, the inverse $(I - A)^{-1} : X \to X$ exists and is bounded. The identity

$$(I - A)^{-1} = I + (I - A)^{-1}A$$

suggests

$$M_n := I + (I - A)^{-1}A_n$$

as an approximate inverse for $I - A_n$. Elementary calculations yield

$$M_n(I - A_n) = I - S_n, \tag{12.12}$$

where

$$S_n := (I - A)^{-1}(A_n - A)A_n.$$

From Lemma 12.9 we conclude that $\|S_n\| \to 0$, $n \to \infty$. Hence for sufficiently large n we have $\|S_n\| \le q < 1$. For these n, by the Neumann series Theorem 3.48, the inverse operators $(I - S_n)^{-1}$ exist and are uniformly bounded by

$$\|(I - S_n)^{-1}\| \le \frac{1}{1 - q}\,.$$

Now (12.12) implies first that $I - A_n$ is injective, and therefore, since A_n is compact, by Theorem 12.1 the inverse $(I - A_n)^{-1}$ exists. Then (12.12) also yields $(I - A_n)^{-1} = (I - S_n)^{-1}M_n$, whence uniform boundedness follows, since the operators M_n are uniformly bounded by Theorem 12.7. The error estimate (12.11) is proven as in Theorem 12.6. □

Note that both error estimates (12.5) and (12.11) show that the accuracy of the approximate solution essentially depends on how well $A_n\varphi$ approximates $A\varphi$ for the exact solution φ.

nice

12.3 Nyström's Method

Recalling Chapter 9, we choose a convergent sequence

$$Q_n(g) = \sum_{k=0}^{n} a_k^{(n)} g(x_k^{(n)})$$

of quadrature formulae for the integral

$$Q(g) = \int_a^b g(x)\,dx$$

with quadrature points $x_0^{(n)}, \dots, x_n^{(n)} \in [a,b]$ and real quadrature weights $a_0^{(n)}, \dots, a_n^{(n)}$. For convenience we write $x_0, \dots, x_n$ instead of $x_0^{(n)}, \dots, x_n^{(n)}$, and $a_0, \dots, a_n$ instead of $a_0^{(n)}, \dots, a_n^{(n)}$. We approximate the integral operator

$$(A\varphi)(x) = \int_a^b K(x,y)\varphi(y)\,dy, \quad x \in [a,b],$$

with continuous kernel K by a sequence of numerical integration operators

the operator approx A_n

$$(A_n\varphi)(x) := \sum_{k=0}^{n} a_k K(x,x_k)\varphi(x_k), \quad x \in [a,b];$$

i.e., we apply the quadrature formulae for $g = K(x,\cdot)\varphi$. Then the solution to the integral equation of the second kind

$$\varphi - A\varphi = f$$

is approximated by the solution of

$$\varphi_n - A_n\varphi_n = f,$$

which reduces to solving a finite-dimensional linear system.

Theorem 12.11 *Let φ_n be a solution of*

$$\varphi_n(x) - \sum_{k=0}^{n} a_k K(x,x_k)\varphi_n(x_k) = f(x), \quad x \in [a,b]. \tag{12.13}$$

Then the values $\varphi_j^{(n)} := \varphi_n(x_j)$, $j = 0, \dots, n$, at the quadrature points satisfy the linear system

$$\varphi_j^{(n)} - \sum_{k=0}^{n} a_k K(x_j,x_k)\varphi_k^{(n)} = f(x_j), \quad j = 0, \dots, n. \tag{12.14}$$

Conversely, let $\varphi_j^{(n)}$, $j = 0, \dots, n$, be a solution of the system (12.14). Then the function $\widetilde{\varphi}_n$ defined by

use tilde in lecture

$$\widetilde{\varphi}_n(x) := f(x) + \sum_{k=0}^{n} a_k K(x,x_k)\varphi_k^{(n)}, \quad x \in [a,b], \tag{12.15}$$

solves equation (12.13).

ie can recon. via Nyström the approx. solns sat. $\varphi_n - A_n\varphi_n$

Proof. The first statement is trivial. For a solution $\varphi_j^{(n)}$, $j = 0, \dots, n$, of the system (12.14) the function φ_n defined by (12.15) has values

$$\varphi_n(x_j) = f(x_j) + \sum_{k=0}^{n} a_k K(x_j,x_k)\varphi_k^{(n)} = \varphi_j^{(n)}, \quad j = 0, \dots, n.$$

Inserting this into (12.15), we see that φ_n satisfies (12.13). □

The formula (12.15) may be viewed as a natural (and exact!) interpolation of the values $\varphi_j^{(n)}$, $j = 0, \dots, n$, at the quadrature points to obtain the approximating function φ_n. It was introduced by Nyström in 1930.

For convenience we note the following analogue of Theorem 12.5.

Theorem 12.12 *The norm of the quadrature operators A_n is given by*

$$\|A_n\|_\infty = \max_{a \le x \le b} \sum_{k=0}^{n} |a_k K(x,x_k)|. \tag{12.16}$$

quadr weights

Proof. For each $\varphi \in C[a,b]$ with $\|\varphi\|_\infty \le 1$ we have

$$\|A_n\varphi\|_\infty \le \max_{a \le x \le b} \sum_{k=0}^{n} |a_k K(x,x_k)|,$$

and therefore $\|A_n\|_\infty$ is smaller than or equal to the right-hand side of (12.16). Let $z \in [a,b]$ be such that

$$\sum_{k=0}^{n} |a_k K(z,x_k)| = \max_{a \le x \le b} \sum_{k=0}^{n} |a_k K(x,x_k)|$$

and choose $\psi \in C[a,b]$ with $\|\psi\|_\infty = 1$ and

$$a_k K(z,x_k)\psi(x_k) = |a_k K(z,x_k)|, \quad k = 0, \dots, n.$$

Then

$$\|A_n\|_\infty \ge \|A_n\psi\|_\infty \ge |(A_n\psi)(z)| = \sum_{k=0}^{n} |a_k K(z,x_k)|,$$

and (12.16) is proven. □

The error analysis will be based on the following theorem.

Theorem 12.13 *Assume the quadrature formulae (Q_n) to be convergent. Then the sequence (A_n) is collectively compact and pointwise convergent (i.e., $A_n\varphi \to A\varphi$, $n \to \infty$, for all $\varphi \in C[a,b]$) but not norm convergent.*

Proof. Since the quadrature formulae (Q_n) are assumed to be convergent, by (9.13) and the uniform boundedness principle Theorem 12.7 there exists a constant C such that the weights satisfy

$$\sum_{k=0}^{n} |a_k^{(n)}| \le C$$

for all $n \in \mathbb{N}$ (see Theorem 9.10). Then we can estimate

$$\|A_n\varphi\|_\infty \le C \max_{a\le x,y\le b} |K(x,y)|\,\|\varphi\|_\infty \tag{12.17}$$

and

$$|(A_n\varphi)(x_1) - (A_n\varphi)(x_2)| \le C \max_{a\le y\le b} |K(x_1,y) - K(x_2,y)|\,\|\varphi\|_\infty \tag{12.18}$$

for all $x_1, x_2 \in [a,b]$. From (12.17) and (12.18) we see that

$$\{A_n\varphi : \varphi \in C[a,b],\ \|\varphi\|_\infty \le 1,\ n \in \mathbb{N}\}$$

is bounded and equicontinuous because the kernel K is uniformly continuous on $[a,b]\times[a,b]$. Therefore, by the Arzelà–Ascoli Theorem 12.3 the sequence (A_n) is collectively compact.

Since the quadrature is convergent, for fixed $\varphi \in C[a,b]$ the sequence $(A_n\varphi)$ is pointwise convergent; i.e., $(A_n\varphi)(x) \to (A\varphi)(x)$, $n \to \infty$, for all $x \in [a,b]$. As a consequence of (12.18), the sequence $(A_n\varphi)$ is equicontinuous. Hence it is uniformly convergent: $\|A_n\varphi - A\varphi\|_\infty \to 0$, $n \to \infty$. That is, we have pointwise convergence: $A_n\varphi \to A\varphi$, $n \to \infty$, for all $\varphi \in C[a,b]$ (see Problem 12.7).

For $\varepsilon > 0$ choose a function $\psi_\varepsilon \in C[a,b]$ with $0 \le \psi_\varepsilon(x) \le 1$ for all $x \in [a,b]$ such that $\psi_\varepsilon(x) = 1$ if $\min_{j=0,\dots,n}|x - x_j| \ge \varepsilon$ and $\psi_\varepsilon(x_j) = 0$, $j = 0,\dots,n$. Then

$$\|A(\varphi\psi_\varepsilon) - A\varphi\|_\infty \le \max_{x,y\in[a,b]} |K(x,y)| \int_a^b \{1 - \psi_\varepsilon(y)\}\,dy \to 0, \quad \varepsilon \to 0,$$

for all $\varphi \in C[a,b]$ with $\|\varphi\|_\infty = 1$. Using this result, we derive

$$\begin{aligned}\|A - A_n\|_\infty &= \sup_{\|\varphi\|_\infty=1} \|(A - A_n)\varphi\|_\infty \ge \sup_{\|\varphi\|_\infty=1}\sup_{\varepsilon>0} \|(A - A_n)(\varphi\psi_\varepsilon)\|_\infty \\ &= \sup_{\|\varphi\|_\infty=1}\sup_{\varepsilon>0} \|A(\varphi\psi_\varepsilon)\|_\infty \ge \sup_{\|\varphi\|_\infty=1} \|A\varphi\|_\infty = \|A\|_\infty,\end{aligned}$$

whence we see that the sequence (A_n) cannot be norm convergent. □

Theorem 12.13 enables us to apply the approximation theory of Theorem 12.10. For the discussion of the error based on the estimate (12.11) we need the norm $\|A\varphi - A_n\varphi\|_\infty$. It can be expressed in terms of the error for the corresponding numerical quadrature by

$$\|A\varphi - A_n\varphi\|_\infty = \max_{a\le x\le b} \left| \int_a^b K(x,y)\varphi(y)\,dy - \sum_{k=0}^{n} a_k K(x,x_k)\varphi(x_k) \right|$$

and requires a uniform estimate for the error of the quadrature applied to the integration of $K(x,\cdot)\varphi$. Therefore, from the error estimate (12.11), it follows that under suitable regularity assumptions on the kernel K and the exact solution φ, the convergence order of the underlying quadrature formulae carries over to the convergence order of the approximate solutions to the integral equation. We illustrate this by the case of the trapezoidal rule. Under the assumption $\varphi \in C^2[a,b]$ and $K \in C^2([a,b]\times[a,b])$, by Theorem 9.7, we can estimate

$$\|A\varphi - A_n\varphi\|_\infty \le \frac{1}{12}\,h^2(b-a) \max_{a\le x,y\le b} \left| \frac{\partial^2}{\partial y^2}\,[K(x,y)\varphi(y)] \right|.$$

Example 12.14 Consider the integral equation

$$\varphi(x) - \frac{1}{2}\int_0^1 (x+1)e^{-xy}\varphi(y)dy = e^{-x} - \frac{1}{2} + \frac{1}{2}\,e^{-(x+1)}, \quad 0 \le x \le 1, \tag{12.19}$$

with exact solution $\varphi(x) = e^{-x}$. For its kernel we have

$$\max_{0\le x\le 1} \int_0^1 \frac{1}{2}\,(x+1)e^{-xy}dy = \sup_{0<x\le 1} \frac{x+1}{2x}\,(1 - e^{-x}) < 1.$$

Therefore, by the Neumann series Theorem 3.48 and the operator norm (12.4), equation (12.19) is uniquely solvable.

We use the (composite) trapezoidal rule for approximately solving the integral equation (12.19) by the Nyström method. Table 12.1 gives the difference between the exact and approximate solutions and clearly shows the expected convergence rate $O(h^2)$.

TABLE 12.1. Numerical solution of (12.19) by the trapezoidal rule

n	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
4	0.007146	0.008878	0.010816	0.013007	0.015479
8	0.001788	0.002224	0.002711	0.003261	0.003882
16	0.000447	0.000556	0.000678	0.000816	0.000971
32	0.000112	0.000139	0.000170	0.000204	0.000243

We now use the (composite) Simpson's rule for the integral equation (12.19). The numerical results in Table 12.2 show the convergence order $O(h^4)$, which we expect from the error estimate (12.11) and the convergence order for Simpson's rule from Theorem 9.8. □

TABLE 12.2. Numerical solution of (12.19) by Simpson's rule

n	$x = 0$	$x = 0.25$	$x = 0.5$	$x = 0.75$	$x = 1$
4	0.00006652	0.00008311	0.00010905	0.00015046	0.00021416
8	0.00000422	0.00000527	0.00000692	0.00000956	0.00001366
16	0.00000026	0.00000033	0.00000043	0.00000060	0.00000086

After comparing Tables 12.1 and 12.2, we wish to emphasize the major advantage of Nyström's method over other methods like the collocation method, which we will discuss in the next section. The matrix and the right-hand side of the linear system (12.14) are obtained by just evaluating the kernel K and the given function f at the quadrature points. Therefore, without any further computational effort we can improve considerably on the approximations by choosing a more accurate numerical quadrature formula.

In the next example we consider an integral equation with a periodic kernel and a periodic solution.

Example 12.15 Consider the integral equation

$$\varphi(t) + \frac{ab}{\pi}\int_0^{2\pi} \frac{\varphi(\tau)d\tau}{a^2+b^2-(a^2-b^2)\cos(t+\tau)} = f(t), \quad 0 \le t \le 2\pi, \tag{12.20}$$

where $a \ge b > 0$. This integral equation arises from the solution of the Dirichlet problem for the Laplace equation in an ellipse with semiaxis a and b (see [39]). Any solution φ to the homogeneous form of equation (12.20) clearly must be a 2π-periodic analytic function, since the kernel is a 2π-periodic analytic function with respect to the variable t. Hence, we can expand φ into a uniformly convergent Fourier series

$$\varphi(t) = \sum_{n=0}^{\infty} \alpha_n \cos nt + \sum_{n=1}^{\infty} \beta_n \sin nt.$$

Inserting this into the homogeneous integral equation and using the integrals (see Problem 12.10)

$$\frac{ab}{\pi}\int_0^{2\pi} \frac{e^{in\tau}d\tau}{(a^2+b^2)-(a^2-b^2)\cos(t+\tau)} = \left(\frac{a-b}{a+b}\right)^n e^{-int} \tag{12.21}$$

for $n = 0, 1, 2, \ldots$, it follows that

$$\alpha_n\left[1+\left(\frac{a-b}{a+b}\right)^n\right] = \beta_n\left[1-\left(\frac{a-b}{a+b}\right)^n\right] = 0$$

for $n = 0, 1, 2, \ldots$. Hence, $\alpha_n = \beta_n = 0$ for $n = 0, 1, 2, \ldots$, and therefore $\varphi = 0$. Now the Riesz Theorem 12.2 implies that the integral equation (12.20) is uniquely solvable for each right-hand side f.

We numerically want to solve (12.20) in the case where the unique solution is given by

$$\varphi(t) = e^{\cos t}\cos(\sin t), \quad 0 \le t \le 2\pi.$$

Using the integrals (12.21), it can be seen that the right-hand side becomes

$$f(t) = \varphi(t) + e^{c\cos t}\cos(c\sin t), \quad 0 \le t \le 2\pi,$$

where $c = (a-b)/(a+b)$.

Since we are dealing with periodic analytic functions, we use the rectangular rule. From Theorem 9.28 we expect an exponentially decreasing error behavior, which is exhibited by the numerical results in Table 12.3 giving the difference between the exact and approximate solutions. Doubling the number of quadrature points doubles the number of correct digits in the approximate solution.

TABLE 12.3. Nyström method for equation (12.20)

	n	$t = 0$	$t = \pi/2$	$t = \pi$
	4	−0.15350443	0.01354412	−0.00636277
$a = 1$	8	−0.00281745	0.00009601	−0.00004247
$b = 0.5$	16	−0.00000044	0.00000001	−0.00000001
	4	−0.69224130	−0.06117951	−0.06216587
$a = 1$	8	−0.15017166	−0.00971695	−0.01174302
$b = 0.2$	16	−0.00602633	−0.00036043	−0.00045498
	32	−0.00000919	−0.00000055	−0.00000069

The actual size of the error, i.e., the constant factor in the exponential decay, depends on the parameters a and b, which describe the location of the singularities of the integrands in the complex plane; i.e., they determine the width of the strip of the complex plane into which the kernel can be extended as a holomorphic function.

Note that for periodic analytic functions the rectangular rule generally yields better approximations than Simpson's rule (see Problem 9.12). □