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MATH 126 WORKSHEET : Green's Theorems

$$\boxed{\text{Recall Div Thm: } \underbrace{\int_\Omega \vec{\nabla}\cdot\vec{a}\, d\vec{x}}_{\text{volume term}} = \int_{\partial\Omega} \underbrace{\vec{n}\cdot\vec{a}\, ds}_{\text{surface flux}}}$$

- Choose $\vec{a} = u\vec{\nabla}v$ where u, v are scalar funcs,

  Expand out the LHS [Hint you may need to guess, or derive a product rule...]

  to get Green's 1st Identity:

- Subtract the above from its $u \leftrightarrow v$ swapped version to get Green's 2nd Identity:

- Choose $v \equiv 1$ & u harmonic to get a surface rule for u:

What is the interpretation?

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~ SOLUTIONS ~

Recall Div Thm: $\displaystyle\int_{\Omega} \vec{\nabla}\cdot\vec{a}\, d\vec{x} = \int_{\partial\Omega} \vec{n}\cdot\vec{a}\, ds$ — volume term; surface flux.

- Choose $\vec{a} = u\vec{\nabla}v$ (grad) where u, v are scalar funcs,

Expand out the LHS [Hint you may need to guess, or derive a product rule...]

to get Green's 1st Identity:

namely $\vec{\nabla}\cdot(u\vec{\nabla}v) = \sum_i \partial_i(u\partial_i v) = \sum_i (u\partial_{ii}v + (\partial_i u)\partial_i v)$
$= u\Delta v + \vec{\nabla}u\cdot\vec{\nabla}v$.

Insert into Div Thm.
so

$$\int_{\Omega} u\Delta v + \vec{\nabla}u\cdot\vec{\nabla}v\, dx = \int_{\partial\Omega} \vec{n}\cdot u\vec{\nabla}v\, ds = \int_{\partial\Omega} u\, \underbrace{\vec{n}\cdot\vec{\nabla}v}_{\text{a.k.a. } \frac{\partial v}{\partial n} := v_n}\, ds$$

- Subtract the above from its $u \leftrightarrow v$ swapped version to get Green's 2nd Identity:

$$\int_{\Omega} u\Delta v + \vec{\nabla}u\cdot\vec{\nabla}v\, dx = \int_{\partial\Omega} u v_n\, ds$$

$$\int v\Delta u + \underbrace{\vec{\nabla}v\cdot\vec{\nabla}u}_{\text{cancels}}\, dx = \int_{\partial\Omega} v u_n\, ds$$

subtract

$$\int_{\Omega} u\Delta v - v\Delta u\, dx = \int_{\partial\Omega} u v_n - v u_n\, ds$$

- Choose $\underbrace{v \equiv 1}_{\text{const.}}$ & $\boxed{u \text{ harmonic}}$ ($\Delta u = 0$) to get a surface rule for u:

$$0 = \int_{\Omega} u\underbrace{\cancel{\Delta v}}_{\text{const}}{}^{0} - v\underbrace{\cancel{\Delta u}}_{\text{harm}}{}^{0} = \int_{\partial\Omega} u\underbrace{\cancel{v_n}}_{\text{const.}}{}^{0} - \underbrace{\cancel{v}}_{\text{const.}}{}^{1}u_n \quad \text{so} \quad \boxed{\int_{\partial\Omega} u_n = 0}$$

What is the interpretation? "harmonic funcs have zero flux though any closed surface"