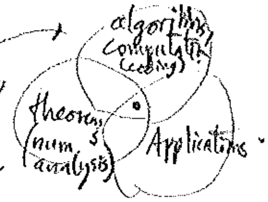
M126 Lec 1 · slips · LNT article. · HW1. ① 1/5/12

welcome - info slips.

Numerical math:



focus on these two.

⇒ your HW is mix of theory & coding numerical methods & experiments. ⟵ practical skills for any math/scientist

Topic: solution of PDEs via integral equations (IEs)

eg. interior BVP: given domain $\Omega \subset \mathbb{R}^2$, cont. func f on $\partial\Omega$, find func $u(x,y)$ on bdry of Ω

$$\begin{cases} \text{st. } \Delta u = 0 \text{ in } \Omega \text{, Laplace's eqn.} \\ \text{ask? } \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} \text{ in } \mathbb{R}^2. \\ \& \quad u = f \text{ on } \partial\Omega \quad \text{boundary data.} \end{cases}$$

→ given $f \in C([0,1])$ ⟵ ask? cont.

$K \in C([0,1]^2)$ ⟵ ask?

func on the square.

Then find func u solving

$$u(t) + \underbrace{\int_0^1 k(t,s)\, u(s)\, ds}_{\text{func of } t\text{: operator acting on } u.} = f(t) \quad \forall t \in [0,1].$$

Fredholm IE "2nd kind".

App: electrostatics (show website), heat steady state, diffusion chemicals.

We will develop some PDE & IE theory along the way

Eg
- uniqueness of BVP is simple. (← max. principle - anyone heard of?)
- but existence is hard: first proven using. IEs (100yrs ago: Fredholm, Hilbert).

The link between BVPs & IEs: instead of $[0,1]$, use the boundary $\partial\Omega$.

Beautiful mathematics & gives efficient num alg! since reduction from 2d to 1d prob! BIE.

We will need other key numerical areas such as rounding errors, quadrature, lin. algebra, convergence...

What is num anal? LNT essay: not just errors, (read from essay). →
- impact of computation huge: 3rd branch of sci.
- NA disasters both. Math, Algs key.
- goal: understand math behind algs, code your own.

30 mins. Syll. (show online). HW → post online, do in Latex. project (hence HW1 due 11 days). X-hrs — coding. eg. meet Wed 3-3:50 do Matlab exercises (please install & go through intro codes)

Languages - doesn't have to be Matlab.

books; we draw from several. Take notes.

Waves? different PDE: $(\Delta + \omega^2) u = 0$ Helmholtz: acoustics, also Maxwell (light, radar) in certain cases.

ω = freq. of waves.

solns. oscillate in space. Why? 1d version $u'' + \omega^2 u = 0$ is ODE w/ solns? $\begin{cases}\sin \omega x \\ \cos \omega x\end{cases}$ ← repeats when $\omega x = 2\pi$ ie $x = \frac{2\pi}{\omega}$ wavelength.

eg show pic on website top.

Comes from wave eqn. for $\tilde{u}(x,y,t)$: $\tilde{u}_{xx} + \tilde{u}_{yy} - \tilde{u}_{tt} = 0$ WE (wave speed = 1).

assume const. freq. $\tilde{u}(x,y,t) = u(x,y)\, e^{-i\omega t}$ sub into WE: $\tilde{u}_{tt} \to (-i\omega)^2 \tilde{u}$

cancel $e^{-i\omega t}$ since holds $\forall t \Rightarrow$ Helmholtz.

Note Laplace & Helm are elliptic PDE (same sign in 2nd derivs), but WE is hyperbolic (1 different sign) → { elliptic: smooth solns; hyperbolic: char. }

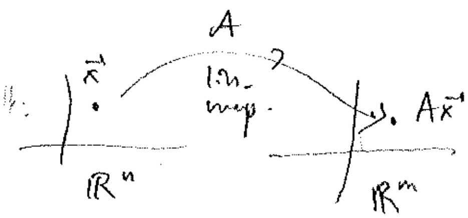
- Scattering: send in a solution to Helm in $\mathbb{R}^2$ (eg. plane wave), but which 'hits' obstacle Ω.

solve exterior BVP $\begin{cases} (\Delta + \omega^2) u = 0 & \text{in } \underbrace{\mathbb{R}^2 \setminus \bar{\Omega}} \\ u = f & \text{on } \partial\Omega . \\ \text{'radiation condition'}. \end{cases}$ ② 1/5/12

? who's seen?
Ω = open set
$\bar{\Omega}$ = closure.
so $\mathbb{R}^2 \setminus \bar{\Omega}$ excludes $\partial\Omega$.

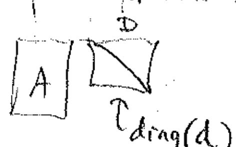
50 mins. break?

Num. Lin. Alg. (Tref + Bau book).

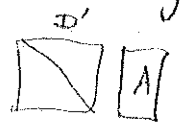
solve $A\vec{x} = \vec{b}$ $\quad A \in \mathbb{C}^{m \times n}$  matrix: 

Review.
mat-vec mult: $A\vec{x}$ is
lin. combo of cols of A w/ coeffs $x_1 \cdots x_n$.

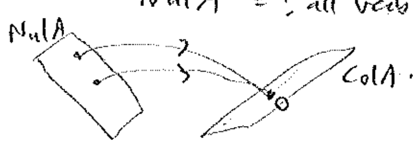
Rescaling columns: what mult A by if want col $\vec{a}_j$ to become $d_j \vec{a}_j$ where $\vec{d} \in \mathbb{C}^n$ given?



postmultiply A by D.

what does  do? rescale rows.

Spaces $\mathrm{Col}\,A = \mathrm{Span}\{\vec{a}_j\} \subset ?\ \mathbb{R}^m$
$\mathrm{Nul}\,A$ = ? all vects which A kills $= \{\vec{x} : A\vec{x} = \vec{0}\} \subset \mathbb{R}^n$

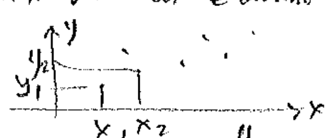


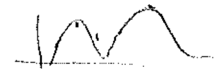
dim Col A called ? rank(A) = # pivots $\le \min(m,n)$

Say A 'tall' $m > n$: when is A (map) one-to-one?
each $\vec{y} = A\vec{x}$ must be unique lin combo of $\{\vec{a}_j\}$ $\Rightarrow$ cols lin. indep $\Rightarrow$ dim Col A = n
Converse also
$\Rightarrow$ Thm ($m \ge n$): A full rank $\Leftrightarrow$ map 1-1 (soln to $Ax = b$ unique, if exists).

Square case $m = n$, (lin. alg). full rank $\Leftrightarrow$ A^{-1} exists (nonsingular). st $A^{-1}A = AA^{-1} = I$. $\Leftrightarrow$ soln $x = A^{-1}b$ exists, unique, for all b.

App: polynomial approx.
Let $\{x_j\}_{j=1}^n$ be distinct set of reals
Claim: $n \times n$ matrix A w/ element $a_{ij} = x_i^{j-1}$ $i,j = 1 \cdots n$ is nonsingular.

How A arise? data $(x_j, y_j)_{j=1}^n$  pts in plane.

What is $n-1$ th degree poly passing exactly thru data? 

$\vec{c}$ = coeffs, $= \{c_0, \cdots c_{n-1}\}$ $\quad p_{\vec{c}}(x) = c_0 + c_1 x + c_2 x^2 + \cdots c_{n-1} x^{n-1}$

match at each pt: Lin. eqns: $p(x_j) = y_j$ ③ 1/5/12

ie $\left\{\begin{array}{l} c_0 + c_1 x_1 + \cdots c_{n-1} x_1^{n-1} = y_1 \\ \vdots \\ c_0 + c_1 x_n \cdots c_{n-1} x_n^{n-1} = y_n \end{array}\right\}$ $A\vec{c} = \vec{y}$

Vandermonde matrix.

Supp. $\vec{c} \neq \vec{c}\,'$ were two such solns,

Then $p_{\vec{c}}(x) - p_{\vec{c}\,'}(x)$ is nontriv. degree-$(n-1)$ poly vanishing at each x_j, ie have n distinct roots $\Rightarrow$ impossible. $\Rightarrow \vec{c}$ unique $\Rightarrow A$ full rank. □

Matlab interlude: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ rank(A) = 2. change $\begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 5 & 10 \end{bmatrix}$ rank = 1. null(A) = $\begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$.
(proj)

n = 10;
~~x = -1 : 0.1 : 1~~ linspace(-1, 1, n)
A = vander(x); A = A(:, end:-1:1) ← reverse order of cols.
A → hard to see #s by eye → ways to view imagesc(A)
spy(A)
plot(A) ← graphs each col.

rank(A) = ~~8~~ 10 as Thm says. ~~but~~ ~~solve~~.
ok try n = 30, 40, 100 : rank(A) never > 36. why?

numerical rank ≠ theor. rank

n = 10; size of soln x ~ 10^2
n = 20 " ~ grows!
→ 10^{16} when size 40 same place!
need singular values of A.

Matlab ...