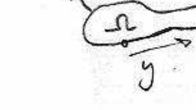
Lec 10 MA26 ① 2/7/12.

elliptic PDE & Potential theory.
(LIE Ch 6)

HW5: eval potential of a bunch of dipoles, check "Gauss' Law".
do Nystrom w/ kernel from this lec.

Last time

$\Omega \subset \mathbb{R}^d$ ($d \geq 2$; we did $d=2$)



$G2^{nd}$ Id: ($u, v \in C^2(\bar\Omega)$)

$$\underbrace{\int_\Omega u\Delta v - v\Delta u \, dx}_{\text{vol.}} = \underbrace{\int_{\partial\Omega} u v_n - v u_n \, ds}_{\text{bdry}}$$

remind $u_n = ?$ $\vec{n} \cdot \vec{\nabla} u$ normal direc-deriv.

$v \equiv 1$, u harmonic? (v const, harm) $0 =$ (const)

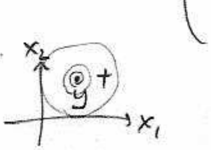
So $\int_{\partial\Omega} u_n = 0$ 'zero flux' (ZF)

Greens Representation Formula: Let $u \in C^2(\bar\Omega)$ be harm. in Ω, then

(GRF) LIE Thm 6.5

$$\int_{\partial\Omega} \Phi(x,y)\, u_n(y) - \frac{\partial\Phi}{\partial n_y}(x,y)\, u(y) \, ds_y = \begin{cases} u(x) & x \in \Omega \quad \text{inside} \\ \frac{1}{2} u(x) & x \in \partial\Omega, \ (\partial\Omega \text{ smooth}) \\ 0 & x \in \mathbb{R}^d \setminus \bar\Omega \quad \text{outside} \end{cases}$$

parse it: $\Phi(x,y)$ Fund sol $= \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x-y|} & \text{in } d=2. \\ \frac{c_d}{|x-y|^{d-2}} \end{cases}$

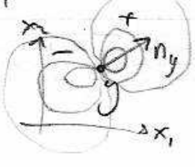


$\Phi(\cdot, y)$ harmonic in $\mathbb{R}^d \setminus \{y\}$

normal directional deriv. wrt. y param: $\frac{\partial\Phi}{\partial n_y} = \underset{\text{normal @ } y}{\vec{n}(y)} \cdot \vec{\nabla}_y \Phi(x,y) \overset{(d=2)}{=} \frac{1}{2\pi} \frac{\vec{n} \cdot (\vec{x} - \vec{y})}{|x-y|^2}$ check

$\partial\Phi(\cdot, y)/\partial n_y$ harm. in $\mathbb{R}^d \setminus \{y\}$.

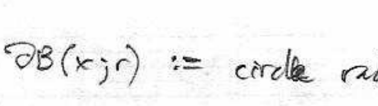
plot vs x: (show Brad)



dipole potential

So its bdry data $u|_{\partial\Omega}$, u_n, enough to reconstruct u everywhere in Ω via a boundary integral!

Pf: $\boxed{\text{case } x \in \Omega}$ ($d=2$, $d>2$ similar) $\partial B(x;r) :=$ circle radius r about x



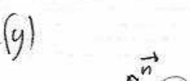
in $R := \Omega \setminus \underbrace{\overline{B(x;r)}}_{\text{closed ball}}$, $\Phi(x,y)$ harm. as func of y.

(ie now x = param, y = coord.)

$\Rightarrow$ $G2^{nd}I$ in R for u, $v = \Phi(x, \cdot)$,

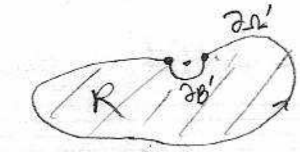
$$0 = \int_R u \underbrace{\Delta_y \Phi(x,y)}_{\text{harm.}} - \Phi(x,y) \underbrace{\Delta u}_{0} = \int_{\partial R = \partial\Omega + \partial B} u(y) \frac{\partial\Phi(x,y)}{\partial n_y} - \Phi(x,y)\, u_n(y)$$

$\longleftarrow$ move $\partial\Omega$ bits ; inwards pointing $\vec{n}$.

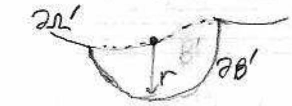


$$\text{So } \int_{\partial\Omega} \Phi(x,y) u_n(y) - \frac{\partial\Phi}{\partial n_y}(x,y)\, u(y) = \int_{\partial B(x;r)} \underbrace{\frac{\partial\Phi(x,y)}{\partial n_y}}_{\frac{1}{2\pi r}} u(y) - \underbrace{\Phi(x,y)}_{-\frac{1}{2\pi}\ln r} u_n(y) \, ds_y$$

$$= \frac{1}{2\pi r} \underbrace{\int_{\partial B} u(y)\, ds_y}_{\text{by MVT } = 2\pi r \cdot u(y) \text{ for some } y \in \partial B} + \frac{1}{2\pi} \ln r \underbrace{\int u_n(y)\, ds_y}_{\text{why? ZF.}} \xrightarrow{\text{take } \lim_{r \to 0}} \lim_{r \to 0} u(y)\Big|_{y \in \partial B} = u(x). \quad \square$$

$\boxed{\text{Case } x \in \partial\Omega}$:  now $\partial R = \partial\Omega' + \partial B'$ where $\partial B' = \partial B(x;r) \cap \Omega$

(sketch)



so in $\lim_{r\to 0}$, $\partial\Omega$ locally flat so $\partial B' \to$ half-circle

$$\& \quad \frac{1}{2\pi r} \int_{\partial B'} u(y)\, ds_y \;\longrightarrow\; \frac{1}{2} u(x)$$

$$\frac{1}{2\pi} \ln r \underbrace{\int_{\partial B'} u_n(y)\, ds_y}_{= \pi r\, u_n(y) \text{ for some } y \in \partial B'} \longrightarrow 0 \quad \text{since } u_n \text{ bnd}, \ \& \ r \ln r \to 0.$$

$\&$ $\partial\Omega' \to \partial\Omega$.

$\boxed{\text{Case } x \in \mathbb{R}^d \setminus \overline{\Omega}}$ There is no ball, $R = \Omega$. $\square$

Useful corollaries

i) since Φ & $\frac{\partial\Phi}{\partial n_y}$ analytic funcs. of 1st var, ie $(x_1, x_2, \ldots)$, u is analytic in Ω wrt. each coord
(LIE Thm 6.6) regardless how nonsmooth bdry data is

ii) mean. val. thm. for harm funcs. (LIE Thm 6.7) if u harm, $u(x) = \frac{1}{2\pi r} \int_{\partial B(x;r)} u(y)\, ds_y$ $(d=2)$
ie val. at center is mean of surface. pf: GRF, bring out $\Phi(x,y)$ const, use ZF. — for $d>2$, it's surf. area

$\Rightarrow$ Maximum principle: harm. funcs attain their max & min. on bdry
pf: let x be isolated interior max, then $\exists\, B(x;r)$, $r>0$ & mean val. $\Rightarrow$ contradiction.

$\Rightarrow$ BVP $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$ has at most one soln. Suppose u_1, u_2 solns, then $w = u_1 - u_2$ harm. with $w|_{\partial\Omega} = 0$, so $w \equiv 0$ in Ω by Max. Princ.

iii) $$\int_{\partial\Omega} \frac{\partial\Phi}{\partial n_y}(x,y)\, ds_y = \begin{cases} -1 & x \in \Omega \\ -1/2 & x \in \partial\Omega \\ 0 & x \in \mathbb{R}^d \setminus \overline{\Omega} \end{cases}$$ pf: GRF w/ $u \equiv -1$ $\left(\begin{array}{l}\text{is harm} \\ u_n \equiv 0\end{array}\right)$

"Gauss' Law" (GL) Can also prove via ZF & small balls directly (try it).

postponed.

Layer potentials (new notation)

$\partial\Omega$ closed curve, "density" $\varphi \in C(\partial\Omega)$, $(S\varphi)(x) := \int_{\partial\Omega} \Phi(x,y)\, \varphi(y)\, ds_y$ are the bdry integrals from GRF.

"single-layer potential"

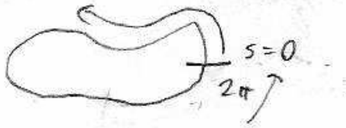
"double-layer pot." $(D\varphi)(x) := \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n_y} \varphi(y)\, ds_y$.

Often use $\begin{cases} \sigma & \text{for SLP density} \\ \tau & \text{DLP} \end{cases}$

eg GRF says, in Ω, $u = S\sigma + D\tau$ where $\sigma = u_n$, $\tau = -u|_{\partial\Omega}$

eg GL says $D1$ generates potential -1 in Ω, 0 outside. $\leftarrow$ test in HW5.

• How eval such integrals in practice? $\left(\substack{d=2\\ \text{case}}\right)$ Change variable:



say $z(s)$ parametrizes $\partial\Omega$, $z(2\pi) = z(0)$ ie $z : [0, 2\pi) \to \mathbb{R}^2$
$(z_1(s), z_2(s))$

eg $\partial\Omega$ given by $f(\theta)$ in polars: $z_1(s) = f(s)\cos s$
$z_2(s) = f(s)\sin s$

Then (if $g : \mathbb{R}^2 \to \mathbb{R}$) $\displaystyle\int_{\partial\Omega} g(y)\, ds_y \overset{\text{change of var.}}{=} \int_0^{2\pi} g(z(s))\, \underbrace{|z'(s)|}_{\text{'speed function'}}\, ds \xrightarrow{\text{quadrature via periodic trap rule}} \frac{2\pi}{N}\sum_{j=1}^{N} g(z(s_j))\, |z'(s_j)|$

At each surface node $z(s_j)$ also need normal

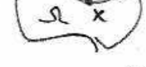
n = "z' rotated CW 90° & normalized"
ie $n(s) = \begin{pmatrix} n_1(s) \\ n_2(s)\end{pmatrix} = \frac{1}{|z'(s)|}\begin{pmatrix} z_2'(s) \\ -z_1'(s)\end{pmatrix}$

Coding: recommend you set up (inline?) funcs $\begin{cases} z(s) \\ z'(s) \\ n(s) \end{cases}$ gives & N then pass the func. handles to routine that:
2) fill a_{ij} matrix els for Nyström
1) plots a potential due to given density func handle $\mu(s)$

Jump relations:

note eg: $\mathcal{D}1$ has jump of +1 in value as x crosses $\partial\Omega$ from inside to outside.
0 -1 $\tau \equiv 1$

Define (let $x \in \partial\Omega$)
$u^\pm(x) := \lim_{h\to 0^+} u(\vec{x} \pm h\vec{n}_x)$ normal at x:



$u_n^\pm(x) := \lim_{h\to 0^+} \vec{n}_x \cdot \vec{\nabla} u(\vec{x} \pm h\vec{n}_x)$

limiting values on $\partial\Omega$ from each side

The expect DLP has $u^+(x) - u^-(x) = \tau(x)$, true:

Thm (JR's) Let $\partial\Omega$ be C^2 (ie $z_1, z_2 \in C^2$), $\sigma, \tau \in C(\partial\Omega)$, and $u = \mathcal{S}\sigma$, $v = \mathcal{D}\tau$, then for $x \in \partial\Omega$,

$$u^\pm(x) = \int_{\partial\Omega} \Phi(x,y)\,\sigma(y)\, ds_y \qquad \text{(no jump in SLP val.)}$$

$$u_n^\pm(x) = \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n_x}\,\sigma(y)\, ds_y \mp \frac{\sigma(x)}{2} \qquad \text{(jump in n. deriv. of SLP)}$$

note!

$$v^\pm(x) = \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n_y}\,\tau(y)\, ds_y \pm \frac{\tau(x)}{2} \qquad \text{(jump in DLP val.)}$$

$$v_n^\pm(x) = \int_{\partial\Omega} \frac{\partial^2\Phi(x,y)}{\partial n_x \partial n_y}\,\tau(y)\, ds_y \qquad \text{(no jump in DLP deriv.)}$$