Lec 11 (M126)

Finish layer plot defn, JKR's last talk.

Bdry integral ops: 

\[ S : C(\mathbb{R}) \to C(\mathbb{R}) \]
\[ \text{has kernel } \phi(y) \text{ – note: weakly singular} \]
\[ D : C(\mathbb{R}) \to C(\mathbb{R}) \]
\[ \text{kernel } \frac{\phi(y)}{dy} \]

So JKR says, \( V^\pm = DT \pm \frac{1}{2} T \)

Say want \( V \) to solve BVP \( Dv = 0 \) in \( \Omega \) \( \neq \) already set by reg. \( V = \partial T \) !

\[ (D - \frac{1}{2}) T = f \]
Fred IE, (which kind? 2nd due to \( \frac{1}{2} \)), a Bdry IE (BIE)

or \( (I - 2D) T = -2f \)
BIE in std. 2nd kind form.

Recipe to solve BVP: i) solve BIE for \( T \), ii) reconstruct \( V = \partial T \) in interior of \( \Omega \).

Then: \( \partial T \) be \( C^2 \) smooth, in \( R^2 \). Then kernel of \( D \) is continuous.

intuitively, \( \{ \partial T \} \) contains of \( \frac{\partial \phi(y)}{dy} \) are circles, local curvature of \( D \) or \( \Omega \) dictates which contour you're on. \( \Rightarrow \) curvature need to be cont. \( \Rightarrow C^2 \).

\[ \text{parametrize by } z : \mathbb{R} \to R^2, 2\pi \text{-periodic. } z \in C^2 \]
\[ \frac{dz}{ds} \text{ means } \frac{d}{ds} \theta \]

wrt \( t, s \in [0, 2\pi] \), kernel \( k_t(s) = \frac{1}{2\pi} \frac{n_t(s) \cdot (t - z(s))}{|z(t) - z(s)|} \)

\( = \) \( \frac{1}{2\pi} \cos \theta \)

Also \( k_t(s) = \frac{1}{2\pi} \cos \theta \)

\[ \lim_{t \to s} k_t(s) \text{ top bot vanish } \Rightarrow \text{Hôpital: } \frac{dt}{ds} \text{ top } = n(s) \cdot \zeta(t) \to 0 \text{ also! Why?} \]

\[ \Rightarrow \frac{d}{ds} \text{ top } = n(s) \cdot \zeta(t) \to n(s) \cdot \zeta(t) \]

\[ \frac{d}{ds} \text{ bot } = 2 \zeta(t) \cdot (t - z(s)) \] \[ \frac{d}{ds} \text{ bot } = 2 \zeta(t) \frac{t}{|z(t) - z(s)|} \to 2 \zeta(t) \frac{t}{|z(t) - z(s)|} \]

Combine: \( \lim_{t \to s} k_t(s) = \frac{1}{2\pi} \frac{n(s) \cdot \zeta(t)}{|z(t) - z(s)|} = -\frac{K}{2\pi} \)

\( K = \text{local curvature} \)

In practice BIE all done wrt \( s \in [0, 2\pi] \), by changing one length \( ds \) to \( |z(s)| ds \)

\( f, \zeta \text{ are funcns on } [0, 2\pi] \), and \( D \) has kernel \( k_t(s) = \frac{1}{2\pi} \frac{n(s) \cdot \zeta(t) - z(s)}{|z(t) - z(s)|} \zeta(s) \)
or on the diagonal, \( k(s, s) = \frac{1}{\Delta t} K(s) \cdot |z(s)| \) in code use \( \frac{v(s) \cdot z''(s)}{|z'(s)|^2} \)

Solving via Nyström \((1 - A)\tilde{v} = -2f\) in sys,

where NNmat \( A \) has entries \( a_{ij} = \epsilon_{ij} + 2K(s_i, s_j) w_j \), \( s_j = \frac{2\pi j}{N} \)

col. vec \( f \) has \( f_j = f(z(s_j)) \) the body data at the nodes.

Solution vector \( \tilde{v} = \{\tilde{v}_j\}_{j=1}^N \) is density at nodes.

Commonly, \( V = D \tilde{v} \) is then approximated using these same quadrature nodes.

...this isn't always accurate!

(Active research by me!)

Thm: above BVP has a soln. Pf: \( D \) kernel cont. \( \Rightarrow D \) cont

Can show \( 2D \) doesn't have \( 1 \) as an eigenvalue. 

\( \Rightarrow \) by Fredholm Alternative, soln. \( \mathcal{C} \) exists \( \Rightarrow \) soln \( v \) exists.

Proof of JK3 (hack): need to show \( v = D \tilde{v} \) can be continuously extended from 

\( \Omega \) to \( \mathbb{R}^4 \).

\( \lim_{y \to z} v_l = v_r \) exist \( \Rightarrow \) \( V \) exist.

1. Split into GL & correction: let \( x = z + \frac{h}{2} \)

\[ v(x) = \mathcal{C} \left( z \right) \int_{\mathbb{R}^N} \frac{2 \delta(x, y)}{\partial \psi (x, y)} ds_y + \int_{\mathbb{R}^N} \frac{2 \psi(x, y)}{\partial \psi (x, y)} \left( \mathcal{C}(y) - \mathcal{C}(z) \right) ds_y \]

by GL = \( \frac{z-x}{h} \) \( h \leq 0 \)

\( \Rightarrow \) vanish as \( y \to z \).

If \( \lim_{h \to 0} v(z, h) = v(z, 0) \), uniformly in \( z \in \mathbb{R}^4 \), we are done, since \( h \) accounts for the jump.

2. Pick radius \( r > 0 \) & split \( y \)-integral into far & local parts:

\[ v(z, h) = v(z, 0) = \int_{|y-z| \geq r} \left( \frac{\partial \psi}{\partial \psi (x, y)} \right) \left( \mathcal{C}(y) - \mathcal{C}(z) \right) ds_y + \int_{|y-z| < r} \left( \frac{\partial \psi}{\partial \psi (x, y)} \right) \left( \mathcal{C}(y) - \mathcal{C}(z) \right) ds_y \]

Assume \( 2h \leq r \), then \( |v(z, h)| \leq 1 \int_{|y-z| < r} \left| \frac{\partial \psi}{\partial \psi (x, y)} \right| |z| ds_y \), so \( |v(z, 0)| \leq C |h| \frac{1}{r^2} \)

\( \frac{1}{r} \leq C \frac{1}{r^2} \)

Lemma (local part): \( \exists h > 0 \) & \( C \) (ind of \( h \)) s.t. \( \int_{|y-z| < r} \left| \frac{\partial \psi}{\partial \psi (x, y)} \right| |z| ds_y \leq C \) & \( 4 |h| \leq h \).
Then \( |v(z) - v(z_0)| \leq C \frac{|z - z_0|}{|z - z_0|^2} + 2C \max_{z \in \mathbb{R}^2} |T(y) - T(z)| \)

\[ \text{given } \varepsilon > 0 \text{ can choose } r > 0 \text{ s.t. } \frac{\varepsilon}{4C} \]

\[ \text{since } T \text{ cont. (}\Rightarrow \text{unif. cont. theorems)} \]

Therefore choose \( h_0 = \frac{\varepsilon r^3}{2C} \)

\[ |v(z) - v(z_0)| \leq \varepsilon \quad \forall z \in \mathbb{R}^2 \quad \text{and } |h| < h_0, \text{ such that } h_0 > 0 \text{ exists for each } \varepsilon > 0 \]

\[ \Rightarrow v(z) \rightarrow v(z_0) \text{ uniformly.} \]

Finally, pf of LIL:

1. **Geometric fact:** "Normal Lemma" (NL):
   - Ex st. \( y(z) \leq L(1 - y_1) \quad \forall y \in \mathbb{R}^2 \)
   - \( y \in \mathbb{R}^2 \)

\[ x = t \quad y = t \]

2. **Laurie's bound on distance** (LBD):
   - \( |x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \)
   - \( x_1 = y_1 + h_1, y_1 = y_2 + h_2 \)
   - \( \Rightarrow \frac{|x - y|}{LBD} \leq \frac{|x - y|}{|x - y|^2} = \frac{1}{LBD} \]
   - \( \frac{1}{LBD} \leq \frac{1}{LBD} \)

\[ \Rightarrow |v(z) - v(z_0)| \leq \frac{1}{LBD} (|z - y|^2 + h_1^2) \quad \forall h_1 < h_0, \quad h_0 > 0 \]

\[ \Rightarrow |v(z) - v(z_0)| \leq C + C \frac{h_1^2}{|z - y|^2 + h_1^2} \]

**Pick patch around z:** see \( n(z) = \nabla n(z) \quad \forall z \in \mathbb{R} \)

**Patch means:** \( \Gamma = \phi_y : |z - y| < r, \ y \in \mathbb{R}^2 \)

Can project onto line \( y = r \) at worst factor \( 2 \) in variable change \( ds \rightarrow ds \)

\[ \Rightarrow |v(z) - v(z_0)| \leq 2C \int_{\Gamma} \frac{b}{s^2 + h_1^2} ds \leq \frac{C}{2} + C \int_{\Gamma} \frac{b}{s^2 + h_1^2} ds \leq C, \quad \text{indep of } \Gamma \]