Lec 11 (M126) • stop at 11:35a! • show singlelayer.m ① 2/9/12

{ finish Layer pot defns, JRs last time.
•

Bdry integral ops: $S: C(\partial\Omega)\to C(\partial\Omega)$ has kernel $\Phi(x,y)$ ← note: weakly singular (diag $y=x \to \infty$).
(S not $\mathcal{S}$)

$\mathcal{D}: C(\partial\Omega)\to C(\partial\Omega)$ kernel $\dfrac{\partial\Phi(x,y)}{\partial n_y}$

So JR3 says, (if $v=\mathcal{D}\tau$) $v^{\pm} = D\tau \pm \frac{1}{2}\tau$

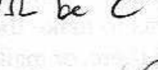
Say want v to solve BVP $\begin{cases}\Delta v = 0 & \text{in } \Omega \leftarrow \text{already sat. by rep. } v=\mathcal{D}\tau\,! \\ v=f & \text{on } \partial\Omega \leftarrow \text{set } v^-=f \text{ \& we've done.}\end{cases}$

$\Rightarrow (D-\frac{1}{2})\tau = f$ Fred. IE, (which kind? 2nd due to ½), a Bdry IE (BIE)

or $\boxed{(I-2D)\tau = -2f}$ BIE in std. 2nd kind form.

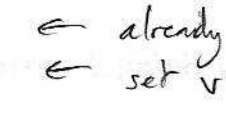
Recipe to solve BVP: i) solve BIE for τ, ii) reconstruct $v=\mathcal{D}\tau$ in interior of Ω.

Thm: let $\partial\Omega$ be C^2 smooth, in $\mathbb{R}^2$. Then kernel of D is continuous.

intuitively,  contours of $\dfrac{\partial\Phi(\cdot,y)}{\partial n_y}$ are circles, local curvature of $\partial\Omega$ dictates which contour you're on. $\Rightarrow$ curvature needs to be cont. $\Rightarrow C^2$.

Pf  parametrize by $z:\mathbb{R}\to\mathbb{R}^2$, 2π-periodic. $\partial\Omega\in C^2$ means $z'(s)=\dot z(s) = \frac{dz}{ds}$, $\ddot z(s) = \frac{d^2z}{ds^2}$ } both cont. rec. funcs of s

Demand $|\dot z|>0$, speed never vanishes.

wrt $t,s\in[0,2\pi)$, kernel $k(t,s) = \dfrac{1}{2\pi}\dfrac{n(s)\cdot(z(t)-z(s))}{|z(t)-z(s)|^2}$ 

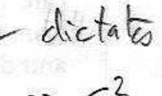
also $k(s,t) = \dfrac{1}{2\pi}\dfrac{\cos\theta}{r}$

top & bot. cont. wrt s & t, $\Rightarrow$ cont. $\forall s\neq t$.

$\lim_{t\to s} k(t,s)$ top & bot vanish $\Rightarrow$ l'Hôpital: $\frac{d}{dt}\text{top} = n(s)\cdot\dot z(t) \to 0$ also! why?

$\Rightarrow \frac{d^2}{dt^2}\text{top} = n(s)\cdot\ddot z(t) \xrightarrow{t\to s} n(s)\cdot\ddot z(s)$

$\frac{d}{dt}\text{bot} = 2\dot z(t)\cdot(z(t)-z(s))$, $\frac{d^2}{dt^2}\text{bot} = 2|\dot z(t)|^2 \to 2|\dot z(s)|^2$

Combine: $\lim_{t\to s} k(t,s) = \dfrac{1}{4\pi}\dfrac{n(s)\cdot\ddot z(s)}{|\dot z(s)|^2} = \dfrac{-\kappa(s)}{4\pi}$ κ = Local curvature $= (\text{curv. radius})^{-1}$ 

• In practice BIE all done wrt $s,t\in[0,2\pi)$, by changing arc length ds_y to $|z'(s)|ds$

$\Rightarrow f,\tau$ are funcs on $[0,2\pi)$, and D has kernel $k(t,s) = \dfrac{1}{2\pi}\dfrac{n(s)\cdot(z(t)-z(s))}{|z(t)-z(s)|^2}\cdot|z'(s)|$ $t\neq s$,
(2π-periodic)

or on the diagonal, $k(s,s) = \frac{-1}{4\pi} \underbrace{\kappa(s)}_{} \cdot |z'(s)|$ ⟶ in code use $-\frac{n(s)\cdot z''(s)}{|z'(s)|^2}$

② 2/9/12

Solving via Nyström $(I-A)\vec{\tau} = -2\vec{f}$ lin. sys,

where $N\times N$ mat. A has entries $a_{ij} = \delta_{ij} + 2k(s_i,s_j)w_j$ $\qquad s_j = \frac{2\pi j}{N}$

col. vec $\vec{f}$ has $f_j = f(z(s_j))$, the bdry data at the nodes. $\qquad w_j = \frac{2\pi}{N} \ \forall j$

Solution vector $\vec{\tau} = \{\tau_j\}_{j=1}^N$ is density at nodes, can be used.

Commonly, $v = \mathcal{D}\tau$ is then approximated using these same quadrature nodes.
interior soln in Ω,

... this isn't always accurate! (active research by me!)

Thm: above BVP has a soln. Pf: $\mathcal{D}$ kernel cont. $\Rightarrow \mathcal{D}$ cpt

Can show $2\mathcal{D}$ doesn't have 1 as an eigvalue.

$\Rightarrow$ by Fredholm Alternative, soln. τ exists $\Rightarrow$ soln v exists

got here.

Proof of JR3 (hard): need to show $v = \mathcal{D}\tau$ can be continuously extended from

Ω to $\overline{\Omega}$ w/ lim. value $v_- = (D - 1/2)\tau$

$v+$, $v-$, Ω

$\mathbb{R}^2\setminus\Omega$ to $\mathbb{R}^2\setminus\Omega$ " $v_+ = (D + 1/2)\tau$

[1] Split into GL & correction: let $x = z + hn_z$ $\quad h, n_z, x, z$

$$v(x) = \tau(z) \underbrace{\int_{\partial\Omega} \frac{\partial\Phi}{\partial n_y}(x,y)\,ds_y}_{\text{by GL} = \begin{cases}-1 & h<0\\ -1/2 & h=0\\ 0 & h>0\end{cases}} + \underbrace{\int_{\partial\Omega} \frac{\partial\Phi}{\partial n_y}(x,y)\big(\tau(y)-\tau(z)\big)\,ds_y}_{=: V(z,h)}$$

note: vanish as $y\to z$.

If show $\lim_{h\to 0} V(z,h) = V(z,0)$, uniformly in $z\in\partial\Omega$, we are done, since GL accounts for the jump.

i.e. continuous wrt h.

[2] Pick radius $r>0$ & split y integral into 'far' & 'local' parts:

$$V(z,h) - V(z,0) = \underbrace{\int_{\substack{y\in\partial\Omega\\ |y-z|\geq r}} \left(\frac{\partial\Phi}{\partial n_y}(x,y) - \frac{\partial\Phi}{\partial n_y}(z,y)\right)\big(\tau(y)-\tau(z)\big)\,ds_y}_{a \quad \text{far}} + \underbrace{\int_{|y-z|<r} \left(\frac{\partial\Phi}{\partial n_y}(x,y) - \frac{\partial\Phi}{\partial n_y}(z,y)\right)\big(\tau(y)-\tau(z)\big)\,ds_y}_{b \quad \text{local}}$$

Assume $2|h|\leq r$, then $|a| \leq 2\|\tau\|_\infty \underbrace{\int_{|y-z|\geq r} \left|\frac{\partial\Phi}{\partial n_y}(x,y) - \frac{\partial\Phi}{\partial n_y}(z,y)\right| ds_y}_{\text{HW: } \leq Ch/r^2}$, so $|a| \leq C\frac{|h|}{r^2}$

↑ dep. on τ, not z.

Lemma (local part): $\exists h_0>0$ & C (indep of h) s.t. $\int_{|y-z|<r} \left|\frac{\partial\Phi}{\partial n_y}(z+hn_z,y)\right| ds_y \leq C \quad \forall |h|\leq h_0$

(L-P)

(pf of LPL to follow.)

③ 2/9/12

Then $|v(z,h) - v(z,0)| \le C\frac{|h|}{r^2} + 2C \cdot \underbrace{\max_{\substack{z\in\partial\Omega \\ |y-z|<r}} |\tau(y) - \tau(z)|}$

given $\varepsilon > 0$ can choose $r > 0$ s.t. $\le \frac{\varepsilon}{4C}$

since τ cont. on $\partial\Omega$ ($\Rightarrow$ unif. cont.)

then choose $|h_0| = \frac{\varepsilon r^2}{2C}$

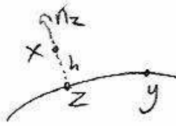
$\Rightarrow |v(z,h) - v(z,0)| \le \varepsilon \quad \forall z \in \partial\Omega$ & $|h| \le h_0$. such an $h_0 > 0$ exists for each $\varepsilon > 0$

$\Rightarrow v(\cdot, h) \longrightarrow v(\cdot, 0)$ uniformly ∎

$\boxed{3}$

Finally, pf of LPL:

2 geom. facts i) "Normal lemma" (NL): ($\partial\Omega \in C^2$) $\exists L$ st. $n_y \cdot (z-y) \le L|z-y|^2 \quad \forall z,y \in \partial\Omega$



in $d=2$ pf is simply that doublelayer kernel is continuous.

$d \ge 2$: Taylor thm wrt. s, param.

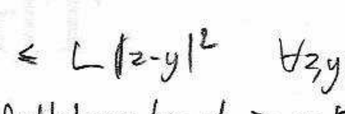
ii) "Lower bnd on distance" (LBD): $|x-y|^2 = |z-y+hn_z|^2 = |z-y|^2 + 2h\underbrace{n_z\cdot(z-y)}_{\substack{(NL) \\ \le L|z-y|^2}} + h^2$

$\ge \frac{1}{2}(|z-y|^2 + h^2) \quad \forall |h| < h_0$, for some $h_0 > 0$

$x-y = z-y+x-z$.

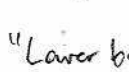
Then $\left|\frac{\partial\Phi}{\partial n_y}(x,y)\right| \le \frac{|n_y\cdot(z-y)|}{2\pi|x-y|^2} + \frac{|n_y\cdot(x-z)|}{2\pi|x-y|^2}$ NL LBD $\sim h$ LBD.

$\le C + C\frac{h}{|z-y|^2+h^2}$



h^2 in denom makes integrable! Cauchy distn

Pick patch Γ_r around z where $n(z)\cdot n(y) \ge 1/2$, $\forall y \in \Gamma_r$.



patch means $\Gamma_r := \{y : |z-y| < r, \ y \in \partial\Omega\}$. Can project onto line w/ at most factor 2 in variable change $ds_y \to ds$

So $\int_{\Gamma_r} \left|\frac{\partial\Phi}{\partial n_y}(x,y)\right| ds_y \le 2C\underbrace{\int_{-r}^{r} 1 + \frac{h}{s^2+h^2}\,ds}_{2r+\pi} \le Cr + C\underbrace{\int_{-\infty}^{\infty} \frac{h}{s^2+h^2}\,ds}_{\pi} \le C$, indep of h

∎