Lec 13 M126 · Colloq today OH 2:30 Th. 2 Fr ① 2/16/12

Laplace variants
Helmholtz, scattering

last time: · int. Dir BVP (proven unique) $\xLeftarrow[u = \mathcal{D}\tau]{u \text{ solves (w/ data } f)}$ $(I - 2D)\tau = -2f$

· int Neu BVP (proven unique up to const, needs $\int_{\partial\Omega} f = 0$) $\xLeftarrow[u = S\sigma]{}$ $(I + 2D^T)\sigma = 2f$

inherits nonuniqueness of BVP, ie $\dim \mathrm{Nul}(I + 2D^T) = 1$

But can solve lin. sys. fine (try it).

These are both "indirect" BIE: pick a representation for soln. u as LP, so that BIE for unknown density comes out 2nd kind

Why not 1st kind? eg try $u = S\sigma$ for int. Dir, want BC
$$S\sigma \underset{JR1}{=} u^- = f$$
but S cpt $\Rightarrow$ ∞ # eigvals accumulating at zero, $\Rightarrow$ ill-conditioned in a bad way (for N large, use iterative rather than $O(N^3)$ direct lin. solvers; they hate such a matrix).

"Direct" BIE also possible: GRF in interior, $x \in \Omega$ then $(S u_n - \mathcal{D} u|_{\partial\Omega})(x) = u(x)$ (*)

take $x \to \partial\Omega^-$ & use JR1 & 3, get $S u_n^- - (D - 1/2)u^- = u^-$
$$\Rightarrow \underbrace{(I + 2D)}\, u^- = \underbrace{S u_n^-}$$
say you want to solve int Neu BVP then $u_n^- = f$, so RHS Sf known

as here, direct gave adjoint of indirect.

the unknown isn't a density, rather, the value.

When BIE solved, use (*) to reconstruct u in Ω.

Note: since we know homog. int. Neu BVP has only const. solns, ie $u^- = \text{const}$, then $\mathrm{Nul}(I + 2D) = \{\text{the const. funcs}\}$

Exterior probs: eg Dirichlet BVP $\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega} \\ u = f & \text{on } \partial\Omega \\ u(x) \text{ bnded as } |x| \to \infty \end{cases}$ ⟵ extra condition needed for uniqueness. (physically: zero total charge on body)

Proof unique soln exists $\forall f \in C(\partial\Omega)$:

let $\tilde{u}(x) := u\left(\frac{x}{|x|^2}\right)$ "Kelvin xform of u", turns outside in, yet $\tilde{u}$ harmonic too, now on
condition "bnded at ∞" becomes "analytic at 0". bnded domain $\Rightarrow$ existence & unique.

Indirect BIE: $u = \mathcal{D}\tau$ (sat bnded at ∞ cond), JR3 gives $(D + 1/2)\tau = u^+ \overset{BC}{=} f$ ie $(I + 2D)\tau = 2f$

signs differ from int Dir BVP, that's all!

expect BIE unique? No! just showed op $I + 2D$ singular.

Worse, BIE has <u>no</u> soln. for certain f, even though BVP <u>does</u> have (unique) soln! (Helmholtz will have similar story)

For, suppose $(I+2D)\tau = 2f$ "complementary BVP haunts the solvability!"

then inner prod $(\phi, (I+2D)\tau) = 2(\phi, f)$ — ie int Neu for ext Dir.

"Ghost of int Neu haunts BIE for ext. Dir.!" the single-layer σ's which generate const funcs

$$= \big(\underbrace{(I+2D^T)\phi}_{\text{zero } \forall \phi \in \mathrm{Nul}(I+2D^T)}, \tau\big) \quad \leftarrow \text{move over}$$

$\Rightarrow$ contradiction unless $f \perp \mathrm{Nul}(I+2D^T)$

(easy part of full version of Fredholm Alternative) Thm 39, Ch.5, Colton.

Literature: various fixes, eg Colton, replace D kernel by $\frac{\partial \Phi(x,y)}{\partial n_y} + 1$ — not Id, rather the "1 kernel"

can prove: exists, unique $\forall f$. (Colton §5.3)

<u>Helmholtz eqn.</u> $\underbrace{(\Delta + k^2)}_{\text{plays role of Laplace op}} u = 0$ $\qquad k = \text{wavenumber} = \frac{2\pi}{\text{wavelength}}$

homog. int. Dir BVP $\begin{cases} (\Delta + k^2) u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

there exist discrete $k_1 < k_2 \le k_3 \le \cdots \to \infty$ s.t. has nontriv. soln

Pf: Δ acting on $\{u \in L^2(\Omega),\ u|_{\partial\Omega} = 0\}$ has <u>cpt</u> inverse — prove since Greens function integral kernel of Δ^{-1}, weakly singular

$\Rightarrow -\Delta u = k^2 u$ has a set discrete 'Dirichlet eigvals' k_j^2, accum only at ∞.

To solve int. Dir BVP $\begin{cases} (\Delta + k^2) u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$ proceed as Laplace, but new kernel; that's it!

kernel: $\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|)$ $\qquad d=2 \qquad H_0^{(1)\prime} = -H_1^{(1)}$

'outgoing' Hankel function, a special func, — see DLMF. Matlab: $\text{besselh}(\nu, z) = H_\nu^{(1)}(z)$

$$\frac{\partial \Phi(x,y)}{\partial n_y} = -\frac{ik}{4} \underbrace{\frac{n_y \cdot (x-y)}{|x-y|}}_{\cos\theta} H_1^{(1)}(k|x-y|)$$

show

Asymptotics: $\Phi(x,y) \underset{x \to y}{=} \frac{1}{2\pi} \log \frac{1}{|x-y|} + O(1)$ ie same singularity as Laplace $\Rightarrow$ same JRs!

large-dist: $H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})} + O(z^{-1})$ as $z \to +\infty$.

Where from? say $u(r,\theta) = f(kr) e^{i\nu\theta}$ polar sep. of var, fix $\nu \in \mathbb{Z}$ & find $f(z)$ st. u sat Helm. Eqn.

$0 = (\Delta + k^2) u = \frac{1}{r}\partial_r(r \partial_r u) + \frac{1}{r^2}\partial_{\theta\theta} u - k^2 u = \left(k^2 f'' + \frac{k f'}{r}\right) e^{i\nu\theta} + (i\nu)^2 \frac{f}{r^2} e^{i\nu\theta} - k^2 f e^{i\nu\theta}$

gather $kr = z$: $z^2 f'' + z f' + (z^2 - \nu^2) f = 0$ Bessel's eqn (ν^{th} order), $H_\nu^{(1)}(z)$ is soln. to ODE w/ certain asymptotics

Ext Dir BVP: for u^s

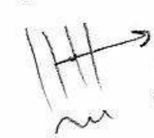
$$(\mathrm{ED}) \left\{ \begin{array}{ll} (\Delta + k^2) u^s = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega} \qquad d = 2, 3, \cdots \\ u^s = f & \text{on } \partial\Omega \\ \lim_{r\to\infty} r^{\frac{d-1}{2}} \underbrace{\left(\frac{\partial u^s}{\partial r} - iku^s \right)}_{\text{ie in } d=2,\ \text{this} = o\left(\frac{1}{\sqrt{r}}\right)} = 0 & \end{array} \right.$$

radiation condition: outgoing (e^{+ikr}) rather than incoming (e^{-ikr}) as $r \to \infty$.

has unique soln. $\forall f \in C(\partial\Omega)$, Colton-Kress Thm 3.7.

Scattering: say 'incident wave' $u^i : \mathbb{R}^d \to \mathbb{C}$, eg. $u^i(x) = e^{ik n_i \cdot x}$

sat $(\Delta + k^2) u^i = 0$ in $\mathbb{R}^d$. plane wave



n_i unit direction vec.

then if u^s solves (ED) w/ $f = -u^i|_{\partial\Omega}$, $u := u^i + u^s$ solves Helm. eqn in $\mathbb{R}^d \setminus \overline{\Omega}$ & $\underbrace{\text{vanishes on } \partial\Omega}_{\text{the physical BC}}$.

why? $u^s|_{\partial\Omega} = f = -u^i|_{\partial\Omega}$ cancelling the inc. wave.

Note: u^i doesn't sat. radiation cond, but new waves due to obstacle (u^s) do.
