Where do Hankel functions come from? \( u \text{ s.t. } (Q, \bar{Q}) \bar{Q}(r, y) = 0 \text{ for } V \text{ everywhere} \) \( y > 0 \).

Call \( u = \bar{Q}(r, y) \), want s.t. Helmholtz.

\[
\begin{align*}
u(r, \theta) &= e^{i\theta} \text{ polarr sol. of var., } r, \theta \in \mathbb{R} \text{ so single-valued, solve for } f : \\
0 &= (\Delta + k^2)u = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + k^2 u = (k f'' + \frac{k f'}{r} + \frac{f'}{r^2} + k^2 f) e^{i\theta} + O(1) \\
\text{cancel } e^{i\theta} \text{ gather } k^2 = z : \\
z f'' + k f' + (z^2 - y) f = 0 \quad \text{Bessel's eqn., order } \nu \end{align*}
\]

Large argument: \( H^{(1)}_\nu(z) = \sqrt{\frac{2}{\pi z}} e^{i (z - \frac{\pi}{2} - \frac{\nu \pi}{2})} + O(\frac{1}{z}) \quad \text{pinned at Collo Q1-Q2} \).

Are also solutions regular at \( r = 0 \): \( J_\nu(z) \) Bessel functions.

By nonuniqueness in 2D, for scattering.

In 1900 you saw out by BIE handled by ghost of complementary BVP: let \( u = \Omega \), solve out BVP if \( \Omega(x + 2D) \bar{Q} = 2 \Omega = -2u \text{ from far field} \).

Huygens' is: \( T \), singular for certain set of \( k \).

\[ H_{\nu}^1(z) \text{ is soln. of log singular } \sim z^{-\nu} \text{ for some } \nu < 0 \]

\[ H_{\nu}^1(z) \text{ is real, } \text{if } \nu \text{ is real} \text{ by J13.} \]

\[ \text{Suppose } \Omega \neq 0 \text{ sat. } (I + 2D) \bar{Q} = 0 \text{ in } \Omega \] \( \Omega = 0 \text{ in } \Omega \) then \( \phi \) is interior Neumann eigenfunction. (acoustic resonance of cavity ||\( u \).)

Then by CRF, \( \Delta \phi_{\nu} - k^2 \phi_{\nu} = 0 \text{ in } \Omega \).

Take \( \nu \rightarrow -2D \) & use J13: \( -\Delta \phi_{\nu} - \phi_{\nu} = 0 \text{ in } (I + 2D) \bar{Q} = 0 \).

Since \( \phi_{\nu} \text{ continuos } \Omega \text{ by CRF}, \text{ dim } \text{Null} (I + 2D) = 0 \text{ singular, not solvable } \).

Shaw equality exist only if \( k^2 \) is \( k^2 \text{ eigenvalue of } \Omega \).

\[ \text{Project: } \text{use small expansion to link such } k^2. \]

Fix \( \nu > 0 \text{ eq. } (\bar{Q} - i\nu S) \bar{Q} = 0 \).

Gradschitz-Weaver, Long, Parish '60's.

\[ \text{Then } (I + 2D - 2i\nu S) \bar{Q} = 0 \] \[ \text{J13 as before } \bar{Q} \text{ s.t. } \bar{Q} \in \Omega \text{ for jump for } \Sigma_{\text{val}}. \]

Then: \( I + 2D - 2i\nu S \) injective \( \forall k > 0 \).

\[ \text{pf: Let } \bar{Q} \text{ solve } (\bar{Q} - D - i\nu S) \bar{Q} = 0 \text{ then } \nu k = 0, \text{ then } \nu k = 0 \text{ by construction of } BIE (2D \bar{Q} = 0). \]
\[ \Rightarrow V = 0 \text{ in } \mathbb{R}^n \text{ by uniqueness of solu. Dir. BVP for potential solns.} \]

\[ \Rightarrow V_n = 0 \text{ on } \partial \Omega \]

\[ \begin{cases} 
V_{1,3} = 0 \\ V_{2,4} = \nabla \cdot V = -i \gamma \varepsilon 
\end{cases} \quad (a) \]

\[ \begin{align*}
\int_{\partial \Omega} \nabla \cdot \nabla V \cdot ds &= \int_{\partial \Omega} \nabla V \cdot \nabla \cdot ds \\
&= -i \gamma \int_{\partial \Omega} \nabla V \cdot ds
\end{align*} \]

Take Im part: \( \gamma = 0 \).  
QED.

Notes:

1) Call this scheme robust since widely used finds; similar exist for Neumann BVP, transmission, etc.

2) Quadrature of BIE near borders: \( S \) has log singularity near diagonal.  
   Approaches: a) use correction of periodic trig. scale weights, so omit diagonal i.e. integrate smooth + log i.e. smooth to high order.
   Kapur-Rokhlin 93/,

   b) find exact weights to integrate log: smooth globally: product quadrature.
   Keast 91, better but more analytic work.

Then also make high order \( \text{H}46 \); 6th order, unlike lagrange 6th degree exponential.

Fast Algorithms:

- eg \( N = 10^6 \): can’t even fill Nyström matrix \( A(10^6 \times 16 \text{ byte} = 16 \text{,000 GB}) \)
  rounded by complex gauss \( \mathfrak{S} \)

  or 2d surface.

  Instead: iterative methods: eg. \text{GMRES}^1 \ (\text{NLA Ch.35}), each it. solve \( x \rightarrow A x \)

  converge, stop when residual order \( \|A x - b\| \) small enough for you.

  For well-conditioned \( 2^{\text{nd}} \)-kind BIE, take only 10-20 iters to get many digits \((10^{-11})\) accuracy.
  But \( 1^{\text{st}} \)-kind terrible convergence rate, useless.

  So now, whole scheme to solve \( x \rightarrow A x \) in \( \mathcal{O}(N^3) \) since \( x \rightarrow A x \) is

  dense \( \mathbb{R}^n \).

Can we apply Nyström method to a vector \( x \rightarrow A x \) faster than \( \mathcal{O}(N^3) \)? Yes!
Let $y_i \in \mathbb{R}^n$ be set of nodes. The problem is to find $A$ has elements $a_{ij} = \frac{1}{\ln |y_i \cdot y_j|}$ if $i \neq j$.

This is the off-diagonal part of Nyström matrix for $\mathcal{S}$ operator (lifting), without weights $w_j$.

In most cases, a low-rank curve is used, with $N = 100$ for small ($\sim 10$) & independent of $N$. The low-rank requires source - target separation.

A low-rank means $\tilde{A} = P Q = \sum_{i=0}^{10} N_i^{10} - 100 \approx$ via SVD (but that's too slow in practice).

Fix an off-diagonal block, call it size $N \times N$: sources $y_i$, $j = 1 \ldots N$, target $z_i$, $i = 1 \ldots N$.

We wish to compute $u_i = \sum_{j=1}^{N} x_j \ln \frac{1}{|x_j - y_j|} = (Ax)_i$, $i = 1 \ldots N$.

Potential due to sources $u(z) = \sum_{j=1}^{N} x_j \ln \frac{1}{|x_j - y_j|}$ harmonic for $z \neq y_j$, $j = 1 \ldots N$.

Goal is eval $u \in$ target $z_i$, $i = 1 \ldots N$.

Thus (multiple expansion), outside a disc $B$ centered at $0$, containing all $y_j$:

$$u(r, \theta) = c_0 \ln \frac{1}{r} + \sum_{n \geq 1} \left( a_n \cos n \theta + b_n \sin n \theta \right) r^{-n}$$

or with $z = re^{i \theta}$, $u(z) = \text{Re} \left\{ c_0 \ln \frac{1}{z} + \sum_{n \geq 1} \left( a_n z^{-n} + b_n z^{-\bar{n}} \right) \right\}$

convergent in $\mathbb{C} \setminus B$.

```latex
\begin{align*}
\end{align*}
```

**Current expansion**