last time: empirically observed low-rank property of off-diagonal blocks, but how explicit this? (without decay in $x,y$, which is $O(\delta)$)

last time: want real potential $u(z) = \frac{1}{|z^j - y^k|}$ at $z = z_i$, $i = 1, \ldots, N$.

for $z \neq y_j, j = 1, \ldots, N$.

$1. R^2$: source taget

$2. R^3$: source target

Note: $u(z)$ harmonic for $z \neq y_j, j = 1, \ldots, N$.

The (multipole expansion): let $B$ be a disc containing all $y_j$, centered at $0$, radius $R$,

then for $r > R$, $u(r, \theta) = c_0 \ln \frac{1}{r} + \sum_{n=1}^{\infty} \left( an \cos \theta + bn \sin \theta \right) r^{-n}$

$\ln$ in polar $\ln$ monopole $\ln$ in polar $\ln$ boundary values at each radius $r$.

Note: solving $c_0 = 0$, true for every finite $u$ harmonic in $r > R$ if $\delta$ is

or considering $R^3 \subset C$, $u(z) = Re \left[ c_0 \ln \frac{1}{r} + \sum_{n=1}^{\infty} c_n z^{-n} \right]$ 

Laplace expansion (Taylor exp. about $0$)

Say truncate to $p$ term, how bad is error?

Consider single point charge $\delta$ at $y$: $u(z) = \ln \frac{1}{z - y}$

let's work w/ complex-valued potential, take $Re$ at end.

$= \ln \frac{1}{z} - \ln (1 - \frac{y}{z})$

Taylor $\ln (1 + x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$ abs. conv. for $|x| < 1$

is multipole expansion, prove above then.

Please truncate error $E_p(z) := \ln \frac{1}{z} + \sum_{n=1}^{p} \frac{y^n}{n} z^{-n} - \ln \frac{1}{z - y} = \sum_{n=p+1}^{\infty} \frac{y^n}{n} z^{-n}$

$p$-term approx.

so $|E_p(z)| \leq \sum_{n=p}^{\infty} \frac{|y|^n}{n} |z|^{-n}$

shift sum.

so $|E_p(z)| = O\left(|\frac{y}{z}|^p\right)$ as $p \to \infty$ exponential conv. in $p$!

Use this bound for each charge in disc $B$, get:

Thus (multipole-exp): potential due to $N$ charges $x_j$, locations $y_j$, inside $B$. $R$ can be rep. by $p$-order multiple expansion in $|z| > b > R$ w/ pointwise error $< C \frac{p}{|y_j|^p} \cdot \frac{|z|^{-p}}{b^p}$.
Use to apply off-diagonal block $X$: 

\[ x^T \hat{b}, \quad \hat{y} \mapsto R \]

Say \( \{z_i\}_{i=1}^m \) well-separated from \( \{y_j\}_{j=1}^N \), i.e.

\[ \text{separation} \]

Recipe

i) Decide \( p \) based on desired accuracy: \( (\delta, \tau) \mapsto \hat{e} \)

\[ \hat{e} \ll 1, \quad \text{eg. } \delta = 10^{-3} \]

ii) Compute multipole series due to sources:

\[ C_s = \sum_{j=1}^N y_j, \quad C_n = \sum_{j=1}^N \frac{y_j}{N} x_j, \quad n = 1 \ldots p \]

iii) Evaluate multipole expansion at targets:

\[ U(\hat{z}_i) = \hat{U}(\hat{z}_i) = C_0 \ln \frac{1}{z_i} + \sum_{n=1}^p C_n z_i^{-n} \quad i = 1 \ldots N \]

Complexity:

\[ \frac{N}{N^2} \sim \frac{N}{N^2} \]

Compare original \( O(N^3) \):

\[ \text{eg. } N = 10^6, \quad p = 10^{-5}, \quad \tau = 10^{-3} \]

Thus speedup is \( 10^{4.5} \approx 30,000 \! \) That's a good algorithm!

Unfortunately, applying mode of $A$, interaction matrix between \( \{y_j\}_{j=1}^N \) \& \( \{z_i\}_{i=1}^m \) is trickier since not all clumps of points well-separated.

Say all \( y_j \)s lie in rectangle, roughly uniformly distributed.

Cover by $M$ square boxes:

\[ \text{box: } B \]

if uniform, \( \approx \frac{N}{M} \) clumps per box.

Effort to get multipole series about each box due to charges in it = \( \sum_{j \in \text{touch box}} \) replaces \( z \) in expansion with \( z_s \) in box center.

Thus, \( y_i = \sum_{j=1}^n A_{ij} x_j \)

\[ \sum_{j \in \text{touch box}} A_{ij} x_j \]

\[ \sum_{j \in \text{touch box}} \]

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\[ \sum_{j \in \text{touch box}} \]

Total effort (\( i = 1 \ldots N \))

\[ = pN + q \frac{N^2}{M} + pMN \]

get coefficients locally, distant.

Optimal here is \( O(pN^2 \sqrt{N}) \)

Fixing \( \delta \), this is \( O\left(\frac{1}{N^2} \right) \)

Too fast: local.

eg. \( N = 10^6, \quad \tau = 10^{-3} \)

\( 10^{4.5} \approx 30,000 \) Ok, but not as great as before.