Last time: $O(N^2)$ alg. for applying interaction matrix $A_i^j$: $O(N^3)$ (old液压 kernel) between N particles
Assuming: uniformly distributed in exchange.

Trick: nearby seen directly.

Relies on a matrix coming from elliptic PDE.

Why is $x \rightarrow Ax$ important to apply? (Krylov methods: apply $A$ repeatedly)

- Enable iterative soln.
- Other apps: compute forces in large, gravitational, fluid (velocity), molecular (electrostatic)

note: methods are either iterative or direct.

Tinny steps to evolve:

- Preparatory work
- Iteration every $\Delta t$ needed...

Bottleneck: each target box has many (M) targets at which many (N) multipole terms have to be evaluated.

Better: combine/Multipole expansion before evaluating at targets in box.

→ 2/28/11 (see next page)

Hierarchical (multilevel) version:

Tree-code:

<table>
<thead>
<tr>
<th>Level 0</th>
<th>Level 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Box 1</td>
<td>Box 2</td>
</tr>
<tr>
<td>Box 3</td>
<td>Box 4</td>
</tr>
</tbody>
</table>

Adaptivity:

- What if not uniform? Subdivide to different levels until $O(N)$ change for box.

Fast multipole method (FMM):

- Gather multipole terms on top level
- Separate local terms on lower levels
- Evaluate local terms on leaves

Bookkeeping: tricky at each level there are boxes not well-sep, for M2L, how to do many child boxes integration lists.

Effort: $O(N)$, Greengard-Kokhlik '87. Also adaptive version. Bottleneck is much smaller!
Where is the bottleneck? If could make smaller box size $L$, less interaction could be done directly. But curiously would have more boxes hence more effort evaluating all their dipole exps at the $(N)$ distant pts!

Need a way to combine dipole exps so all target pts in a box can be evaluated from single expansion… a "local expansion" = Taylor expansion.

Say $z_0 \in \mathbb{C}$ is source box center, rep. by dipole. @ $z_0$, $|z_0| > 2R$, then can be rep. by Taylor

Consider terms in dipole,

eg monopole $\ln \frac{1}{z-z_0} = \ln \frac{1}{z_0} - \ln \left(\frac{z}{z_0} - 1\right) - \ln (x-y, \bar{z})$ for l.c.1.

$$\ln \frac{1}{z_0} = \frac{1}{z_0} - \frac{1}{2z_0} z + \frac{1}{2z_0^2} z^2 - \frac{1}{3z_0^3} z^3 + \ldots$$

$N^m$-pole $(z-z_0)^m$ has

0th Taylor coeff: $c_0 = \frac{(z-z_0)^m}{z_0} \big|_{z_0} = (-z_0)^m = (-1)^n z_0^{-m}$

1st coeff: $c_1 = \frac{\partial}{\partial z} \bigg|_{z_0} (z-z_0)^m = -n (z-z_0)^{m-1} \bigg|_{z_0} = -n (-z_0)^{-m-1} = (-1)^n z_0^{-m-1}$

$\cdots$

$m$th coeff: $c_m = \frac{1}{m!} \bigg|_{z_0} \frac{\partial^m}{\partial z^m} (z-z_0)^m = \frac{1}{m!} (-1)^n n(n-1) \cdots (n+m-1) z_0^{-m-n}$

So,

Thm (M2L, "multipole to local") : dipole exp, $u(z) = c_0 \ln \frac{1}{z-z_0} + \sum_{n=1}^\infty c_n (z-z_0)^{-n}$ can be written as Taylor expansion $\sum_{n=0}^\infty a_n z^n$, abs. convergent in $|z| < |z_0|$, with coeffs

$$a_0 = c_0 \ln \frac{1}{z_0} + \sum_{n=1}^\infty \frac{c_n}{n} z_0^{-n}$$

$$a_n = c_n (-)^n \frac{n+1}{n!} z_0^{-n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{c_n}{n-1} z_0^{-n-m} , m = 1, 2, \ldots$$

Thm (error of M2L): if source $\mathcal{S}_1, \mathcal{S}_2$ lie in $|z_0| < R$, $|z_0| > b+R$, for some $b>R$, then error of truncating above sum to $p$ terms is, in $|z| < R$, bounded by $\mathcal{C}(\sum_{j=1}^p |x_j|) (R)^p$.

Proof: Green's-Rachlin '87. Same exponential convergence rate as before.

For cial. can now becomes: for each target box compute all coeffs due to each dipole src box $s_{2m}$, evaluate local (Taylor) exp at all targets in the target box. Effort is $O(p^2 M^2)$ since $p^2$ to map $c_n$'s to $a_m$'s, $M^2$ translation $z_0$ (many are) + $O(p N)$ eval. $p^2$-order local exp at all $N$ target pts.

Total effort now $p N + \frac{9 N^2}{M} + p^2 M^2 + p N \quad \text{balance}, \quad M = N^{1/2}$

Overall scaling $O(p^2 N^{3/2})$

$= O(N^{3/2})$ if fixed $p$. Best yet. Can do even better w/ hierarchical version: PAIM.