lec 5. M126. · WS · Runge applet. · demos: charges — potential, etc ① 1/19/12

Write top of WS on board.?)
WS on n=1 lagrange.

Equispaced nodes bad: Demo applet on smooth func. $f(x) = \frac{1}{1+25x^2}$ (C^∞, ie all derivs. cont.) $a=-1$, $b=1$

$x_j = -1 + \frac{2j}{n}$ in $[-1,1]$ (show Runge applet). → interpolant L_nf { converges as $n\to\infty$ in central part, blows up in outer regions of $[-1,1]$! as $n\to\infty$. why?

But if cluster pts near ends,
e.g. $x_j = -\cos\frac{j\pi}{n}$ 'Chebyshev nodes', uniformly convergent, ie $\max_{x\in[-1,1]}|f - L_nf| \to 0$ as $n\to\infty$.
why?

If assume nothing about nodes, product $\left|\prod_{j=0}^{n}(x-x_n)\right| \le (b-a)^{n+1} =: H^{n+1}$ (length of interval.)

& then $\|f - L_nf\|_\infty \le \frac{\|f^{(n+1)}\|_\infty}{(n+1)!} \cdot H^{n+1}$ $\quad \|f-L_nf\|_\infty$ "L^∞ bound".

We want small! $\quad$ max Taylor coeff of f expanded at any point in (a,b)

How big are high Taylor coeffs of a func?

Recall Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n = \frac{f^{(n)}(0)}{n!}$ (about 0) : std result [If f analytic at 0, $\exists \rho > 0$ st. series converges absolutely $\forall z$ in disc $|z|<\rho$ & diverges $\forall z$ outside, $|z| > \rho$, & f analytic in $|z|<\rho$.

abs. conv. ⇒ for $|z| = \rho - \varepsilon$, $\forall \varepsilon > 0$: terms decreasing as $n\to\infty$ certainly ⇒ $|a_n|(\rho-\varepsilon)^n \le C$

ie $|a_n| \le \frac{C}{(\rho-\varepsilon)^n}$

Let's say f analytic in open domain containing 'stadium' $\{z\in\mathbb{C} \mid \text{dist}(z,[a,b]) \le R\}$ (analytic, a, b, R), then Taylor coeffs $|a_n| \le \frac{C}{R^n}$ unif. w.r.t. expansion center in $[a,b]$

then $\|f - L_nf\|_\infty \le \frac{C}{R^n} H^{n+1} \le C\left(\frac{H}{R}\right)^n \to 0$

if $R > H$, exponentially fast as $n\to\infty$.

So if f analytic in (a, b, H, >H) get good uniform convergence regardless of nodes.
much stronger than merely C^∞

But if f analytic in neighborhood of $[a,b]$, but singularities are H or closer, may fail to converge. (Runge).

I leave for you to say where poles of $\frac{1}{1+25x^2}$ are! (a, b, pole.)

$\boxed{\text{The bad news}}$: if construct sq. of interp. operators (L_n) each with $\{x_j^{(n)}\}_{j=0}^{n}$ nodes,

Thm (8.17) (Faber): for each such seq. $\exists f \in C[a,b]$ st. $L_nf \not\to f$ unif. on $[a,b]$.

$\boxed{\text{The good news}}$: Thm (8.16, Marcinkiewicz) For each $f\in C[a,b]$, $\exists$ sq of nodes st. $L_nf\to f$ unif on $[a,b]$

Why best to cluster nodes at ends of $[-1,1]$? [skip]? (Trefethen, Spec. Meth §5)

$$\frac{1}{n+1} \ln \Big| \underbrace{\prod_{j=0}^{n}(z-x_j)}_{=:\, q_{n+1}(z)} \Big| = -\frac{1}{n+1}\sum_{j=0}^{n} \ln\frac{1}{|z-x_j|} = \text{electrostatic potential in } \mathbb{R}^2=\mathbb{C}$$

due to $n+1$ charges strength $\frac{-1}{n+1}$ at nodes. $=: \phi_{n+1}(z)$

so $|q_{n+1}(z)| = e^{(n+1)\phi(z)}$

Say as $n\to\infty$, nodes tend to fixed density func $\rho(x)>0$ on $[-1,1]$, then $\phi_{n+1} \to$

normalized $\int_{-1}^{1}\rho(x)\,dx = 1$. $\qquad \phi(z) = -\int_{-1}^{1}\rho(x)\ln\frac{1}{|z-x|}\,dx$

Uniform case $\rho = 1/2$ so $\phi(z) = \frac{1}{2}\int_{-1}^{1}\ln|z-x|\,dx$: $\to$ you can eval.

& check $\phi(0) = -1$ but $\phi(\pm 1) = -1+\ln 2$ larger at ends.

so $|q_{n+1}|$ is $\approx e^{(n+1)\ln 2} = 2^{n+1}$ times bigger at ends

- Show charges_potential.m.

$\nwarrow$ this same applies to $\ell_k(x)$: for $k \approx n/2$ in middle. exp. large at ends $\Rightarrow$ unstable.

Is there a ρ density that gives ϕ, hence $|q_n|$, const on $[-1,1]$?

- show charges_equilibrate.m

Can show via complex analysis (map from exterior of disc to line); $\rho(x) = \frac{1}{\pi\sqrt{1-x^2}}$ Chebyshev density.

This is density that $x_j = -\cos\frac{\pi j}{n}$ approach  

Gives smallest poss $\max_{z\in[-1,1]} \phi(z)$ hence smallest $|q_{n+1}|$, best interp. convergence.

Can show singularities of f (analytic on $[a,b]$) can be arb. close to $[a,b]$ & still get exponential conv.

$\hookrightarrow$ spectral method?

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§9.1 Quadrature.

want to approximate $Q(f) := \int_a^b f(x)\,dx$

use $Q_n(f) := \sum_{k=0}^{n} \overset{\text{weights}}{w_k}\, f(\overset{\text{nodes in } [a,b]}{x_k})$.

$\Big\}$ $Q, Q_n : C[a,b] \to \mathbb{R}$, linear functionals.

Given nodes, what are good weights? Pick st. $Q_n(f) = \int_a^b (L_nf)(x)\,dx$ ie integrate the interpolation poly exactly $\Rightarrow$ 'interpolatory' quad.

$$= \sum_{k=0}^{n} \underbrace{\int_a^b \ell_k(x)\,dx}_{\text{fixed}\ \to w_k}\ f(x_k)$$

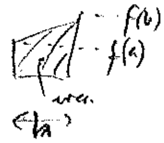
Thm (9.2) given distinct nodes $\{x_j\}_{j=0}^n$, the above $\{w_j\}_{j=0}^n$ are the unique set which integrate all $p\in\mathbb{P}_n$ exactly.

Pf: $Q_n(p) = \int_a^b (L_np)(x)\,dx = \int_a^b p(x)\,dx$ exact. Unique since $\sum w_k f(x_k) \overset{\text{since } f=L_nf \text{ @ nodes}}{=} \sum w_k (L_nf)(x_k) \overset{\text{exact}}{=} \int (L_nf)(x)\,dx \Rightarrow$ interpolatory

so interp $\Leftrightarrow$ exact.

So exact integration up to degree n can be taken as defining feature: called Newton–Cotes (1600's) (sometimes used to mean equispaced)

(Eg. n=1) $w_0 = \int_a^b \ell_0(x)\,dx = \int_a^b \frac{x-b}{a-b}\,dx = \frac{1}{2}(b-a) = \frac{h}{2}$, w_1 = same   ③ 1/19/12

so $Q_1(f) = \frac{h}{2}\left(f(a)+f(b)\right) =$  trapezoid rule.

Error anal? Thm 9.4 Let $f \in C^2[a,b]$, then $\underbrace{\int_a^b f(x)\,dx - Q_1(f)}_{E_1(f)} = -\frac{h^3}{12} f''(\xi)$ for some $\xi \in [a,b]$

pf. $E_1(f) = \int_a^b f(x) - L_1 f(x)\,dx = \int_a^b \underbrace{(x-a)(x-b)}_{\le 0} \underbrace{\frac{f(x)-L_1 f(x)}{(x-a)(x-b)}}_{\text{cont. by l'Hopital at endpoints.}} dx$

MVT for integrals: if $g \geq 0$, $f \in C$, then $\int_a^b g f\,dx = g(z)\int_a^b f\,dx$ for some z in (a,b)

so $E_1(f) = \underbrace{\frac{f(z) - L_1 f(z)}{(z-a)(z-b)}}_{\text{by Thm 8.10 last time}} \underbrace{\int_a^b (x-a)(x-b)\,dx}_{= -\frac{1}{6}h^3}$

$= \frac{f''(\xi)}{2!}$ some ξ.