Claim: 2n-1 is highest pass degree for (n+1)-node quad. 
\[ \int_{-1}^{1} (x-x_j)^2 \, dx = \frac{2}{3} \quad \therefore \beta_j = \frac{2}{3} \quad \alpha_j = \frac{\sqrt{3}}{2} \]

The (18) Gauss weights non-negative, \[ \frac{1}{\beta_j} \int_{-1}^{1} (x-x_j)^2 \, dx = \frac{2}{3} \]

Then for periodic quad, convergent (last line).

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\[ \frac{Q_n(p)}{Q(p)} = \alpha_n \quad \text{for Gauss quad and orders n rule error} \quad \text{eq.} \quad \frac{1}{2} \int_{-1}^{1} f^{(2n+2)}(x) \, dx \quad (2n+2)! \]

Periodic Quadrature §9.4.

Let \( f \in C^{2m+1}[0, \pi] \) be 2\( m \)-periodic, then
\[ \left| Q_n(f) - Q(f) \right| \leq C \int_{-1}^{1} |f^{(2m+2)}(x)| \, dx \cdot \frac{1}{n^{m+1}} \]

If \( f \in C^{m+1} \), then error is \( O(n^{-m}) \) for each \( m > 0 \), called super-algebraic convergence.

But if \( f \) analytic, is even better: exponential conv.

\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \]

* First review complex: \( f(z) \) holomorphic in \( D \subseteq C \): means analytic at each pt., in \( D \).

E.g., \( \frac{1}{1-z^2} \) holomorphic in \( D \setminus \{ 0, \pm 1 \} \).

Simple poles: \( f(z) = \frac{b}{z-a} \) res. due.

Residue Thm.: if \( f \) holomorphic in \( D \) apart from finite \# poles,
\[ \int_{\partial D} f(z) \, dz = 2\pi i \sum \text{residues for each pole}. \]

\( \times \) log of thin (strokes).

May also derive from trigonometric interpolation, i.e. Fourier series truncated at term \( n \frac{1}{2} \), is also exp. accurate.
Converse of this holds: Lemma 9.14: if \( \exists j \) node sat. \( q_j = 0 \) \( \iff \) it's a Gauss quad.

\[ \text{pf: recall interpolatory quad. has } S_w f(x_k) = \int_a^b \frac{f(x)}{\prod_j (x-x_j)} \, dx \quad \forall f \in C([a,b]) \]

claim each \( p \in P_n \) can be written \( p = L_n p + q \) for some \( q \in P_{n-1} \)

\[ \text{why? } p - L_n p = 0 \text{ at } \{x_j\}, \text{ so } q \text{ can have at most } (2n+1) - (n+1) = n \text{ zeros}. \]

\[ S_w p(x) = \int_a^b (L_n p(x)) \, dx + \int_a^b q(x) \, dx = \sum_i w_i p(x_i) \quad \Box \]

So, if can find \( q = 1, x, x^2, \ldots \) are lin. ind. on \([a,b]\), so Gram-Schmidt unique:

\[ q_0 = 1, \quad q_1 = x - \frac{\langle x, q_0 \rangle}{\langle q_0, q_0 \rangle} q_0, \quad q_2 = x^2 - \frac{\langle x^2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 - \frac{\langle x^2, q_0 \rangle}{\langle q_0, q_0 \rangle} q_0, \quad \ldots \]

in this way you'll prove this can be done via 3-term recurrence, i.e. qn involves qk & qk only.

Legendre poly's (but above not std normalization): unique seq. of orthog poly's on \([a,b]\)

with unweighted inner product \( \langle f, g \rangle \).

Lemma 9.16 \( q_n \) has \( n \) simple zeros all in \([a,b]\) (good, so they give a Gauss quad!)

\[ \text{pf: } A_n = 1, \quad q_n - q_0 \text{ is } S_q = 0 \text{ so } q_n \text{ has } \geq 1 \text{ zeros } x_1 \ldots x_n \text{ in } [a,b] \]

Supp. \( m < n \), then \( r_m = \prod_{j=1}^m (x-x_j) \in P_{n-m} \) so is \( + q_m \) contradiction.

but \( \int_{r_m} q_n = 0 \text{ since } r_m q_n \text{ has fixed sign, not } 0. \) \( \Box \)

In practice, how compute \( e_j \)? They are eigenval of \( \left[ \begin{array}{c} 0 & \beta_1 \\ \beta_1 & 0 & \beta_2 \\ \vdots & \vdots & \ddots & \ddots \end{array} \right] \) tridiagonal matrix 

\[ \beta_n = \frac{1}{2\sqrt{1 - \beta_{n-1}^2}} \]

This is \( O(n^3) \) slow! Göller, Lin-Robbins, Grau's better.

See code gauss.agm.