Periodic numerical quadrature

The simplest rule to approximate $\int_{0}^{2\pi} f(t) \, dt$ is sometimes the best: sum $N$ equally spaced samples of $f$!
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The simplest rule to approximate \( \int_0^{2\pi} f(t) dt \) is sometimes the best: sum \( N \) equally spaced samples of \( f \) !

**Theorem (Davis ’59):** Let \( f \) be \( 2\pi \)-periodic, and *real analytic*, meaning \( f(z) \) is bounded and analytic in some strip \( |\text{Im } z| \leq a \) of half-width \( a > 0 \). Then there is a const \( C > 0 \) (indep. of \( N \)) such that the error is

\[
\left| \frac{2\pi}{N} \sum_{j=1}^{N} f \left( \frac{2\pi}{N} j \right) - \int_0^{2\pi} f(t) dt \right| \leq Ce^{-aN}
\]

- exponential convergence in \( N \): doubling \( N \) squares your accuracy
  - very desirable: can get accuracies of \( 10^{-14} \)
  - w/ little effort. Carries over to solving the PDE!
Proof

Residue Thm: \[ 2\pi i \sum \text{residues} \ = \text{closed contour integral in } \mathbb{C} \]
**Proof**

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Beautiful cotangent function \(\cot(z)\): poles at \(\pi j, j \in \mathbb{Z}\), residues 1

- \(\cot(z)\) tends to \(-i\), exponentially fast

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*Image: A diagram showing a contour integral and its pole locations.*
Proof

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Beautiful cotangent function \(\cot(z)\): poles at \(\pi j, j \in \mathbb{Z}\), residues 1

\(f\) analytic \[\frac{1}{2i} f(z) \cot\left(\frac{N}{2}z\right)\): poles at \(\frac{2\pi}{N} j\), residues \(\frac{1}{iN} f\left(\frac{2\pi}{N} j\right)\)
Proof

Residue Thm: \[ 2\pi i \sum \text{residues} = \text{closed contour integral in } \mathbb{C} \]

Beautiful cotangent function \( \cot(z) \): poles at \( \pi j, j \in \mathbb{Z} \), residues 1

\[ f \text{ analytic} \quad \frac{1}{2i} f(z) \cot\left(\frac{N}{2} z\right): \text{poles at } \frac{2\pi}{N} j, \text{ residues } \frac{1}{iN} f\left(\frac{2\pi}{N} j\right) \]

Res. Thm in strip: \[ \frac{2\pi}{N} \sum_{j=1}^{N} f\left(\frac{2\pi}{N} j\right) = \int_{\Gamma_1+\Gamma_2} \frac{1}{2i} f(z) \cot\left(\frac{N}{2} z\right) dz \]
Proof

Residue Thm: \( 2\pi i \sum \text{residues} = \text{closed contour integral in } \mathbb{C} \)

Beautiful cotangent function \( \cot(z) \): poles at \( \pi j, j \in \mathbb{Z} \), residues 1

\[
\cot(z) \text{ tends to } -i, \text{ exponentially fast}
\]

\[
f \text{ analytic} \quad \frac{1}{2i} f(z) \cot\left(\frac{N}{2} z\right) \quad \text{poles at } \frac{2\pi}{N} j, \text{ residues } \frac{1}{iN} f\left(\frac{2\pi}{N} j\right)
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Res. Thm in strip: \[
\frac{2\pi}{N} \sum_{j=1}^{N} f\left(\frac{2\pi}{N} j\right) = \int_{\Gamma_1+\Gamma_2} \frac{1}{2i} f(z) \cot\left(\frac{N}{2} z\right) \, dz
\]

integrand pure \( \text{Im} \) on \( \mathbb{R} \), so
Re parts antisymmetric \( \uparrow \) add
Im parts symmetric \( \uparrow \) cancel

\[
= \text{Re} \int_{\Gamma_1} (-i) f(z) \cot\left(\frac{N}{2} z\right) \, dz
\]
\[
\frac{2\pi}{N} \sum_{j=1}^{N} f \left( \frac{2\pi}{N} j \right) = \text{Re} \int_{\Gamma_1} (-i) f(z) \cot \left( \frac{N}{2} z \right) dz
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\frac{2\pi}{N} \sum_{j=1}^{N} f \left( \frac{2\pi}{N} j \right) = \text{Re} \int_{\Gamma_1} (-i) f(z) \cot \left( \frac{N}{2} z \right) \, dz
\]

Cauchy integral formula in \( D_1 \) (since \( f \) analytic):

\[ - \int_{\Gamma} f(z) \, dz = \int_{\Gamma_1} f(z) \, dz \]
\[
\frac{2\pi}{N} \sum_{j=1}^{N} f \left( \frac{2\pi}{N} j \right) = \text{Re} \int_{\Gamma_1} (-i)f(z) \cot \left( \frac{N}{2} \frac{z}{z} \right) dz
\]

Cauchy integral formula in $D_1$ (since $f$ analytic):

\[
- \int_{\Gamma} f(z) \, dz = \int_{\Gamma_1} f(z) \, dz
\]

add Re part of this to previous eqn:

\[
\frac{2\pi}{N} \sum_{j=1}^{N} f \left( \frac{2\pi}{N} j \right) - \int_{\Gamma} f(z) \, dz = \text{Re} \int_{\Gamma_1} \left[ 1 - i \cot \left( \frac{N}{2} \frac{z}{z} \right) \right] f(z) \, dz
\]