Periodic numerical quadrature

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Theorem (Davis '59): Let f be 2π-periodic, and *real analytic*, meaning $f(z)$ is bounded and analytic in some strip $|\,\mathrm{Im}\, z| \le a$ of half-width $a > 0$. Then there is a const $C > 0$ (indep. of N) such that the error is

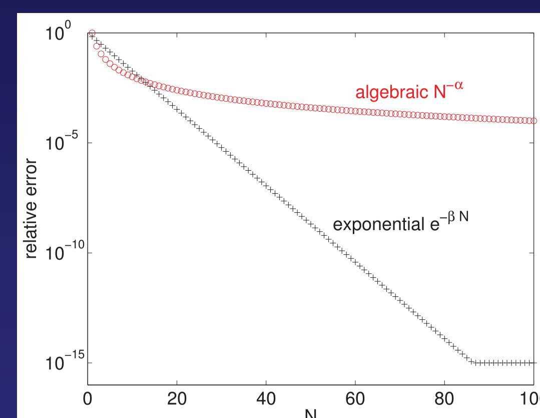
$$\left| \frac{2\pi}{N} \sum_{j=1}^{N} f\left(\frac{2\pi}{N} j\right) - \int_0^{2\pi} f(t)dt \right| \le Ce^{-aN}$$

- exponential convergence in N:

doubling N squares your accuracy

very desirable: can get accuracies of 10^{-14}

w/ little effort. Carries over to solving the PDE!



Proof

Residue Thm: $2\pi i \sum \text{ residues } = \text{closed contour integral in } \mathbb{C}$

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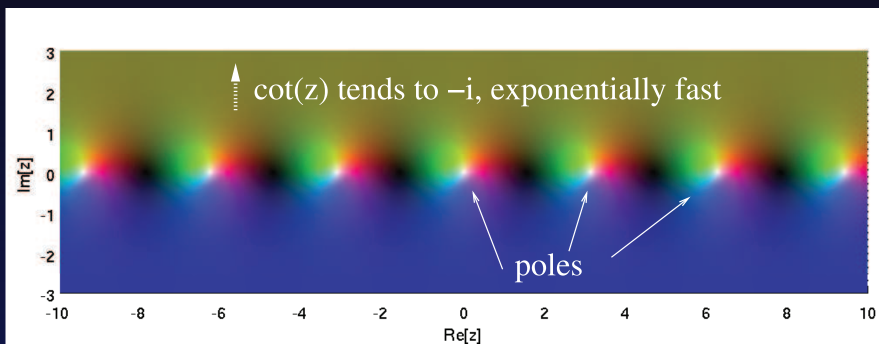
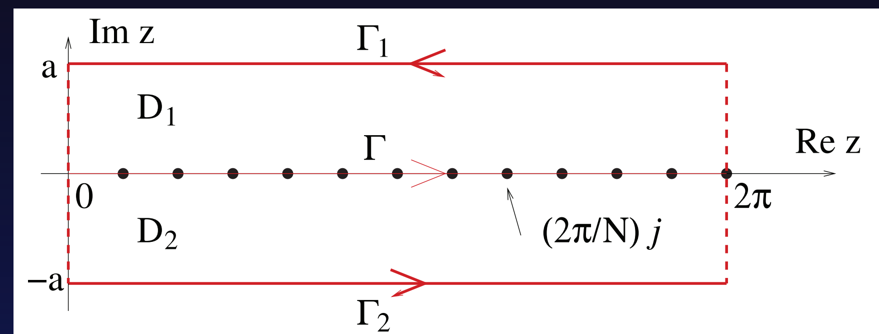
Beautiful cotangent function $\cot(z)$: poles at πj, $j \in \mathbb{Z}$, residues 1





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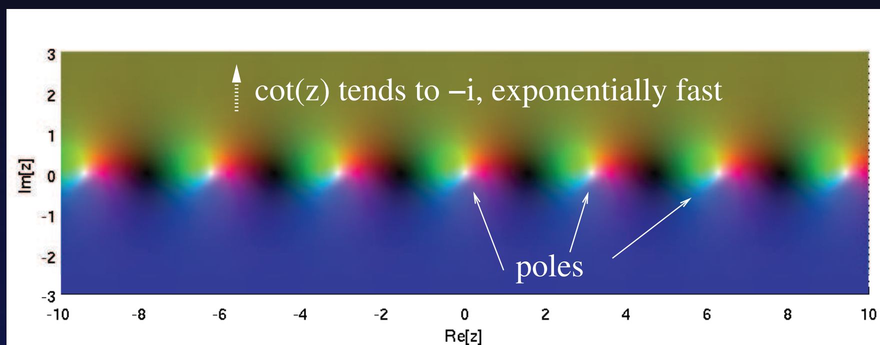

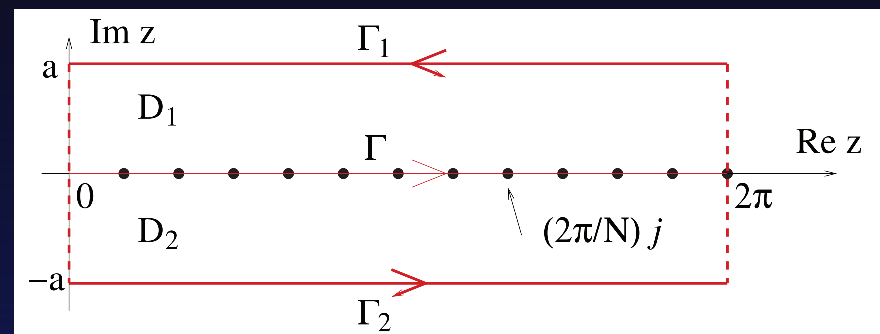

f analytic $\frac{1}{2i} f(z) \cot(\frac{N}{2} z)$: poles at $\frac{2\pi}{N} j$, residues $\frac{1}{iN} f(\frac{2\pi}{N} j)$

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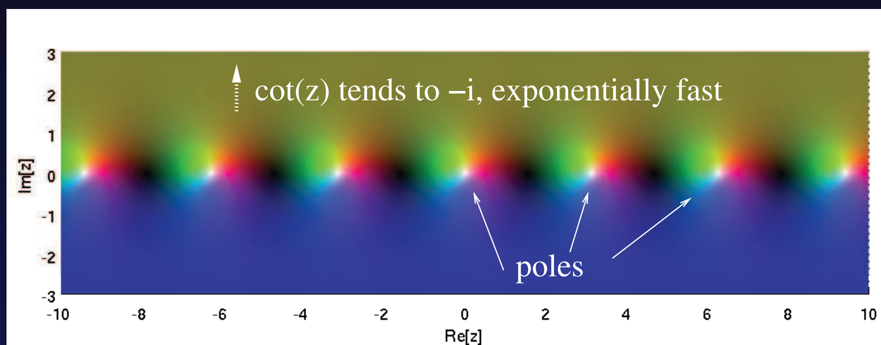

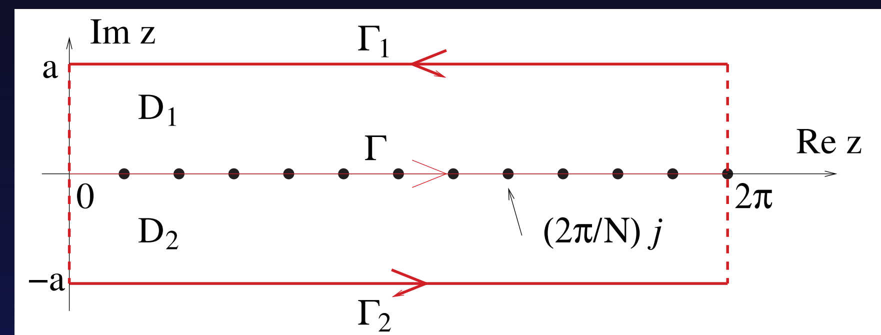

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Res. Thm in strip: $$\frac{2\pi}{N} \sum_{j=1}^{N} f\left(\frac{2\pi}{N} j\right) = \int_{\Gamma_1+\Gamma_2} \frac{1}{2i} f(z) \cot\left(\frac{N}{2} z\right) dz$$

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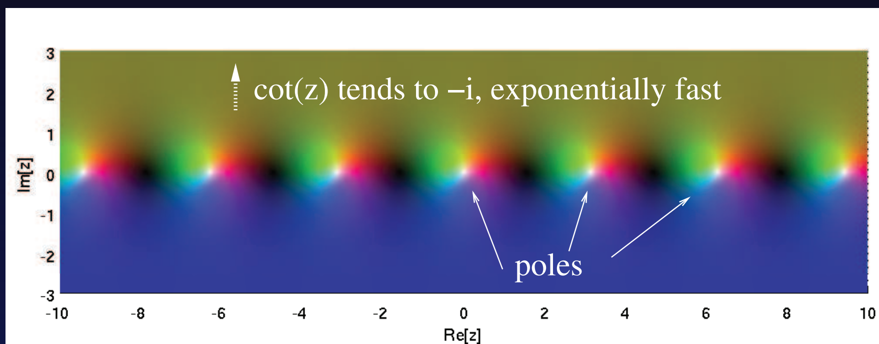

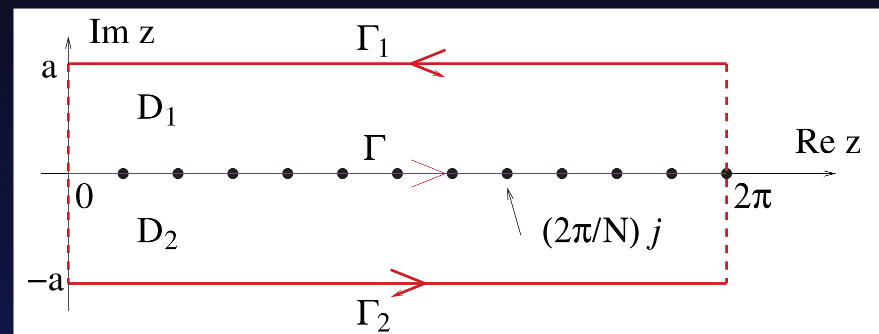

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integrand pure Im on $\mathbb{R}$, so

Re parts antisymmetric $\updownarrow$ add

Im parts symmetric $\updownarrow$ cancel

$$= \operatorname{Re} \int_{\Gamma_1} (-i) f(z) \cot\left(\frac{N}{2} z\right) dz$$

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Cauchy integral formula in D_1 (since f analytic):

$$-\int_{\Gamma} f(z)dz \;=\; \int_{\Gamma_1} f(z)dz$$

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add Re part of this to previous eqn:

$$\frac{2\pi}{N}\sum_{j=1}^{N} f\left(\frac{2\pi}{N} j\right) - \int_{\Gamma} f(z) dz \;=\; \mathrm{Re}\int_{\Gamma_1} \left[1 - i \cot(\frac{N}{2} z)\right] f(z) dz$$