Integral Equations: given interval \([a,b]\), some \(f\) on \([a,b]\), ker \( F\) on \([a,b]^2\),

solve \( \int_a^b k(t,s)u(s)\,ds = f(t) \quad \forall t \in [a,b] \)

Fredholm 1st kind. \( (Ku)(t) = f(t) \quad \forall t \in (a,b) \), right-hand side.

or 2nd kind \( u(t) + (Ku)(t) = f(t) \quad \forall t \in (a,b) \).

Fredholm eqn: \( Ku = f \)

visualise like \( A\mathbf{x} = \mathbf{b} \).

What is \((K^2u)(t)\)? \( \int_a^b \int_a^b k(t,s)k(s,r)u(r)\,dr\,ds \)

when \( k' \) is kernel of \( K^2 \)

So \( k'(t,r) = \int_a^b k(t,s)k(s,r)\,ds \)

[like inner prod. \( \langle A\mathbf{v},\mathbf{w}\rangle = \sum_i a_{ij}v_j w_j \).]

* if \( k(t,s) = 0 \) for \( s > t \) \( \Rightarrow \) inner triangular, thus \( \forall t, \) Volterra, not Fredholm.

\( \mathbf{u} \) can be written \( \sum_{t=0}^{\infty} k(t,s)u(s)\,ds = f(t) \)

has unique soln. \( K \) exists and \( K \) is invertible.

eg. \( K = 1 \): \( \sum_{t=0}^{\infty} u(s)\,ds = f(t) \iff u(t) = f'(t) \)

Fredholm has still on both sides of deg: \( \int_a^b t^2 \,u(s)\,ds = \frac{t^3}{3} \quad 0 < t < 1 \).

\( \Rightarrow \) t-2 \( \int_0^t u(s)\,ds = \frac{t^2}{3} \quad \text{so} \quad \int_0^t u(s)\,ds = \frac{t^3}{3} \quad \text{is a soln.} \)

so \( \text{soln. highly non-unique} \), typ. 1st kind.

\( K \) is rank-1 since for any \( \mathbf{u} \), \( (K\mathbf{u})(t) = \text{a multiple of } t \).

Bounded operator: \( \|K\| = \sup_{\|\mathbf{u}\| = 1} \|K\mathbf{u}\| \quad \forall \text{norms} \).

Eqn space = \( (a,b) \) w/ \( \mathbf{u} \)-norm:

\( \|K\| = \sup_{(t,s) \in (a,b)^2} \|K(t,s)\| = \sup_{t \in (a,b)} \|K(t,\cdot)\| = \sup_{s \in (a,b)} \|K(\cdot,s)\| \) \( \text{if } \| \mathbf{u} \| = 1 \) \( \Rightarrow \) \( \|K\| \) is defined.

eg. \( K_{1/2} \) has \( K_{1/2} > \infty \).

Can say more: above \( \|K\| \) is \( = 1 \). Why? \( \text{Continuous} \Rightarrow \text{bounded} \) \( \Rightarrow \text{approximate} \) \( \text{linear} \).

This is the Volterra norm. (Explicitly gives example of this.)
Kernel may blow up on diagonal, eg \( k(t, s) = \frac{1}{t-s} \)

But if \( |k(t, s)| \leq \frac{c}{|t-s|^{1+\varepsilon}} \) \( \forall t, s \), \( 0 < \varepsilon \leq 1 \) the \( L^1 \) norm of each column bounded,
\( \Rightarrow ||k||_{L^1} \leq C \) \( \text{bounded} \)

called 

'weakly singular'. \( \varepsilon \) strongly singular, may be unbounded operator.


\[ u(t) = \int_0^t k(t, s) u(s) \, ds = f(t) \quad \text{for } t \in \Omega \]

approx \( u \) by \( u_n \) which satisfy

\[ u_n(t) = \sum_{j=1}^n w_j k(t, s_j) u_n(s_j) = f(t) \quad (n) \]

\( \text{or} \quad (I - K_n) u_n = f \)

Then values at nodes \( u_n(s_i) := u_n(s_i) \) set: the (nu) syst.

\[ V_i = 1, \ldots, u_n(s_i) = \sum_{j=1}^n w_j k(s_i, s_j) u_n(s_j) = f(s_i) \quad (LS) \]

ie \( (I - A)^{-1} u_n(s_i) = f(s_i) \quad \) \text{vector } \quad \text{RHS at nodes vector}.

Solve \( (I - A)^{-1} u_n(s_i) \) for \( u_n(s_i) \) gives (nu) syst.

Thm (12.11) If any vector \( \{ u_i^{(n)} \}_{i=1}^n \) is soln. to \( (LS) \) then \( u_n(t) = f(t) + \sum_{j=1}^n w_j k(t, s_j) u_j^{(n)} \)

\( \Rightarrow \text{call formula } (N) \text{ for Nyström interp.} \)

Ps: \( u_n(s_i) = u_n^{(i)} v_i \) where set \( t = s_i \in \Omega \) gives \( (LS) \)

Use this to sub for \( u_j^{(n)} \) in \( (N) \) turns it into \( (x), V_i. \) Subtle!

\( x \) expresses \( u, \) so \( f \) is span \& column slice of kernel at node? \( k(s_i, s_j) \), form interpolation basis.

(20) is equiv. of Vandermonde sys., requiring interp. at nodes \( \Rightarrow (N) \) interp. formula.

if \( \Omega \) is 1D, can apply to 1st kind, but here is no interpolation formula \( (N) \) new, just get \( f(s_i), \) \( k(s_i, s_j) \). Note: \( ||K_n - K|| \to 0 \) as \( n \to \infty \) not converged in norm topology. What sub will conver

\[ \left\| (K_n - K) \phi \right\| \to 0 \] as \( n \to \infty \) (can still prove stuff?)

But do have pointwise convergence.