Compact operators - just the essentials: (see [18], [19], [21])

- may have non-empty range but behave like finite-dim ops (ie, square matrices).

\[ X = C[0,1] \text{ topological space}, \quad f \in X \text{ is a point in } X. \]
Choose metric norm eg, \( \| f \| \). 

- \( \{ f_n \}_{n=1}^\infty \) bounded if \( \| f_n \| \leq C \quad \forall n \leq 1, \ldots \quad \) not seq. geometrically.
- \( \{ f_n \}_{n=1}^\infty \) converges to \( f \in X \) if \( \forall \varepsilon > 0 \) no matter how small, \( \exists N \) s.t. \( \| f_n - f \| < \varepsilon \quad \forall n \geq N \).

Then (Bolzano-Weierstrass) if \( \text{dim}(X) < \infty \), every bounded seq. contains a convergent subseq.

\[ f \in f_1 \in f_2 \in f_3 \in \cdots \]

eg \( X = \mathbb{R} \) - only way to avoid some limit pt is the escape to \( \pm \infty \).

But \( \infty \)-dim spaces such as \( C([0,1]), L^1([0,1]) \) all limits f st \( \int f \, dx < \infty \).

eg Fourier seq \( \{ \sin nx \}_{n=1}^{\infty} \) bounded in \( L^1(0,1) \) but has no convergent subseq.: \( \sqrt{n} \) mutually orthogonal.

Defn: linear op. \( K : X \rightarrow Y \) between normed lin. spaces \( X, Y \) is compact if given any bounded seq. \( \{ f_n \} \) in \( X \), the seq. \( \{ Kf_n \} \) contains a convergent subseq.

- Eg. if \( K \) has finite-dim range \( \mathbb{R}^N : B_W \subseteq (Kx_n) \) has convex. subseq. \( \Rightarrow K \text{ comp.} \).
- But K-Id \( X \rightarrow X \) : can feed it Fourier seq. \( \Rightarrow K \text{ not comp.} \).

Useful facts:
- Ept. op. maps unit ball to hyperellipsoid w/ axes maxes shrink to zero.

\[ 1) \text{ Ept. ops have finite eigenvalues w/ zero the only limit: } K \Phi = \lambda \Phi \text{ then } \lim_{\lambda \to 0} \lambda = 0. \]
\[ 2) \text{ Ept. op. is bounded (easy to prove).} \]

3) Integral operators w/ weakly singular kernel, \( |K(t,s)| \leq \frac{C}{t-s} \), \( t \neq s \), is ept. in \( L^1 \) s.t. is ept. in \( L^2(0,1) \).  

4) \( K \) ept. if it is the operator norm limit of seq. \( K_1, K_2, \ldots \) of ept. op., ie \( \lim_{n \to \infty} \| K_n - K \| = 0 \).

eg acting on sequence \( K \{ a_1, a_2, \ldots \} = \{ K(a_1), K(a_2), \ldots \} \), see \( u_n = 0. \text{ Then can truncate to finite-dim ops } K_n \).

5) Fredholm Alternative: Let \( K : X \rightarrow X \) be ept.

Then (either i) for each \( f \in X \), \( (I-K)x = f \) has unique soln. \( u \in X \)

\[ \text{ or ii) homogeneous } (I-K)x = 0 \text{ has nontrivial soln. } \Rightarrow x \neq 0 \text{ is any soln.} \]

This asserts existence of soln to 2nd kind IE from uniqueness... amazing.

Behave like finite linear systems: \( Ax = 0 \) has soln. \( u \) iff \( Ax = 0 \) has only the trivial soln. \( u = 0 \).

6) \( K \) comp. => convergence rate of Nyström method for 2nd kind IE is \( \| u_n - u \| \leq C \| K u - K u \| \).

It's same rate as quadrature scheme applied to \( K(t,r)u(r) \).
Lec 9 (M126) part 2: PDEs.

Laplacian \( \Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \) in \( \mathbb{R}^2 \) length \( |x| = 1 \) open bounded domain \( x = (x_1, x_2) \)

\[ \Delta u = 0 \quad \text{in } \Omega \quad \Rightarrow u \text{ harmonic in } \Omega \quad (\Leftrightarrow u = \Re v \text{ for some } v \text{ analytic in } \Omega \cap \mathbb{R}^2) \]

Check \( \ln \frac{1}{|x|} = -\ln |x| \) obeys \( \Delta \ln \frac{1}{|x|} = 0 \quad \forall x \neq 0 \)

\[ \frac{\partial}{\partial x_1} \ln |x| = \frac{1}{2} \frac{x_1}{x_1^2} \ln (x_1^2 + x_2^2) = \frac{1}{2} \frac{x_1}{x_1^2} \left( x_1^2 + x_2^2 \right) = \frac{x_1}{x_1^2} \]

\( \Rightarrow \) Fundamental Soln. \( \Phi(x, y) := \frac{1}{2\pi} \ln \frac{1}{|x-y|} \) obeys \( \Delta \Phi(x, y) = 0 \quad \forall x \neq y \)

(Note: already seen in quantum stuff in \( C = \mathbb{R}^3 \)) \( C \) shifts the spike to sit at \( y \).

Divergence Thm: \( \vec{a} = (a_1, a_2) \) vector field (eg \( a_1, a_2 \in C^1(\Omega) \) if sing may have removed)

Thus \( \int \vec{a} \cdot d\vec{x} \text{ vol.} = \int \vec{n} \cdot d\vec{s} \text{ surf. length measure on } \partial \Omega \)

\[ \int \vec{a} \cdot d\vec{x} = \int \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} \quad \Rightarrow \text{ Flux} = \int_{\partial \Omega} \vec{n} \cdot (\vec{a} \, d\vec{s}) \]

\( \Rightarrow \) W.L.S.

(Choose \( \vec{a} = u \vec{i} + v \vec{j} \) where \( u, v \) scalar, funce.)

\( \Rightarrow \) prod rule \( \nabla \cdot (u \vec{i} + v \vec{j}) = u \Delta v + \nabla v \cdot \nabla v \) check.

\[ \text{curl, grad, "screw flow" (2F) } \int_{\partial \Omega} \vec{n} \cdot d\vec{s} = 0 \quad \text{note } \vec{n} \cdot \nabla u = \text{ Un normal deriv} \]

Directional deriv of Fund Soln. \( \text{say } \vec{n} \text{ is a unit vector } \vec{i} \)

deriv of \( \Phi(x, y) \) with moving source pt. \( y \) in \( \vec{n} \) direction

\[ \frac{\partial}{\partial y} \ln \frac{1}{|x-y|} = -\frac{1}{2} \frac{\partial}{\partial y} \ln (x-y)^2 = \frac{1}{2} \frac{x_1 y_2 - x_2 y_1}{(x-y)^2} = \frac{1}{2} \frac{(x_1 y_2 - x_2 y_1)}{(x-y)^2} = \frac{x_1 y_2 - x_2 y_1}{(x-y)^2} \]

So \( \frac{\partial \Phi}{\partial y} = \frac{1}{2\pi} \frac{\vec{n} \cdot (\vec{x} - \vec{y})}{(x-y)^2} \)

is harmonic for \( x \neq y \).