Lec 6 (1) 2/2/12

Compact operators — just the essentials: (see [IE], [NA] Ch-12)

↳ may have ∞-dim range but behave 'like' finite-dim ops (ie, square matrices).

$X = C[a,b]$ topological space, $f \in X$ (func.) is a 'point' in X. Choose metric norm eg. $\|f\|_\infty$.

- seq $(f_n)_{n=1,2,\dots}$ bnded if $\|f_n\| \le C \quad \forall n = 1,2,\dots$ note: seq. goes forever, a long time!
- seq (f_n) converges to $f \in X$ if $\forall \varepsilon > 0$ no matter how small, $\exists N$ st $\|f_n - f\| < \varepsilon \quad \forall n \ge N$.

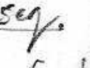
Thm (Bolzano-Weierstrass) if $\dim(X) < \infty$, every bnded seq. contains a convergent subseq.

$f_0\ \underline{f_1}\ f_2\ \underline{f_3}\ \underline{f_4}\ f_5\ f_6\ \underline{f_7}\ \dots$

subseq also goes on forever!

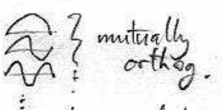
eg $X = \mathbb{R}$: the only way to avoid some limit pt is to escape to $\pm\infty$.

But ∞ dim spaces such as $C[a,b]$, $L^2(a,b)$ ← all funcs f st $\int_a^b |f|^2 dx < \infty$.

eg Fourier seq $\{\sin nx\}_{n=1}^{\infty}$ bnded in $L^2(0,\pi)$ but has no convergent subseq:  } mutually orthog.

Defn: linear op. $K: X \to Y$ between normed lin. spaces X, Y is compact if given any bounded seq. (f_n) in X, the seq (Kf_n) contains a convergent subseq.
↳ "feels" finite dim.

- Eg. if K has finite-dim range $\mathbb{R}^N$: $BW \Rightarrow (Kx_n)$ has conv. subseq $\Rightarrow K$ cpt.
- But $K = Id$ in ∞-dim space $Y = X$: can feed it Fourier seq $\Rightarrow K$ not cpt.

Useful facts: (prove later?) Cpt op. maps unit ball to hyperellipsoid w/ successive semiaxes shrinking to zero: 

1) Cpt ops have discrete eigenvalues w/ zero the only limit: $K\phi = \lambda\phi$ then $\lim_{j\to\infty} \lambda_j = 0$ (intuition.)

2) Cpt $\Rightarrow$ bnded (easy to prove)

3) integral operator on $[a,b]$ w/ {continuous or weakly singular} kernel, $|k(t,s)| \le \frac{C}{|t-s|^\gamma}$, $\gamma < 1$, $\forall s,t$, is cpt (in ∞- or 2-norm)

4) K cpt if: it is the operator norm limit of seq $K_1, K_2, \dots$ of cpt op, ie $\lim_{n\to\infty} \|K - K_n\| = 0$ (cute!)

eg acting on sequences, $K\{a_1, a_2, \dots\} := \{\mu_1 a_1, \mu_2 a_2, \dots\}$ consts. Say $\mu_n \to 0$. Then can truncate to finite-dim ops K_n
↳ $\|K\|_\infty$
& $(K - K_n)\{a_1, a_2, \dots\} = \{0, \dots, 0, \mu_{n+1} a_{n+1}, \dots\}$
so $\|K - K_n\| \to 0$. & K is cpt.

5) Thm (Fredholm Alternative). Let $K: X \to X$ be cpt

Then either i) for each $f \in X$, $(I - K)u = f$ has unique soln $u \in X$

or ii) homog. eqn $(I - K)u = 0$ has nontrivial soln (ie, $\lambda = 1$ is eigval of K)

This asserts existence of soln to 2nd kind IE from uniqueness... amazing!

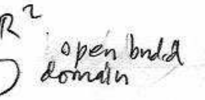
Behave like finite linear systems: $A\vec{x} = \vec{b}$ has soln. $\forall \vec{b}$ iff $A\vec{x} = \vec{0}$ has only the triv. soln (nonsingular)

6) K cpt $\Rightarrow$ convergence rate of Nyström method for 2nd kind IE is $\|u_n - u\|_\infty \le C\|Ku - K_n u\|_\infty$

see [NA Ch.12]. Ie same rate as quadrature scheme applied to $k(t,\cdot)u(\cdot)$.

Lec 9 (M126) part 2: PDEs. ② 2/2/12

Laplacian $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ in $\mathbb{R}^2$. $x = (x_1, x_2)$



$= \text{div grad}$

$\Delta u = 0$ in Ω $\Leftrightarrow$ u harmonic in Ω ($\Leftrightarrow$ $u = \mathrm{Re}\, v$ for some v analytic in $\Omega \subset \mathbb{C} \simeq \mathbb{R}^2$)

Check $\ln\frac{1}{|x|} = -\ln|x|$ obeys $\Delta \ln\frac{1}{|x|} = 0$ $\forall x \neq 0$.

eg $\frac{\partial}{\partial x_1}\ln|x| = \frac{1}{2}\frac{\partial}{\partial x_1}\ln(x_1^2+x_2^2) = \frac{1}{2(x_1^2+x_2^2)}2x_1 = \frac{x_1}{|x|^2}$

etc.

$\Rightarrow$ Fundamental Soln. $\Phi(x,y) := \frac{1}{2\pi}\ln\frac{1}{|x-y|}$ ($x \in \mathbb{R}^2$, $y \in \mathbb{R}^2$) obeys $\Delta_x \Phi(x,y) = 0$ $\forall x \neq y$

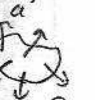
shifts the 'spike' to sit at loc. y.

(Note: already seen in quadrature stuff in $\mathbb{C} \simeq \mathbb{R}^2$)

Divergence Thm: $\vec{a} = \begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix}$ vector field (eg $a_1, a_2 \in C^1(\Omega)$, Ω may have corners).

then $\int_\Omega \underbrace{\vec{\nabla}\cdot\vec{a}}_{\text{div}\,\vec{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2}}\, dx$ (vol.) $= \int_{\partial\Omega} \vec{n}\cdot\vec{a}\, ds$ (surface (arclength measure on $\partial\Omega$))

$\leftarrow$ 'flux' $= \int_{\partial\Omega} \vec{n}_y \cdot \vec{a}(y)\, ds_y$



$\to$ W/S.

(choose $\vec{a} = u\vec{\nabla}v$ where u, v scalar funcs).

& prod rule $\vec{\nabla}\cdot(u\vec{\nabla}v) = u\Delta v + \vec{\nabla}u\cdot\vec{\nabla}v$ check.

G1I, G2I, "zero flux" (ZF) $\int_{\partial\Omega} u_n\, ds = 0$. note $\vec{n}\cdot\vec{\nabla}u =: u_n$ normal deriv.

directional deriv. of Fund. Sol: say $\vec{n}$ is a unit vector ↗

deriv. of $\Phi(x,y)$ wrt moving source pt. y in $\vec{n}$ direction: $\frac{\partial\Phi(x,y)}{\partial n_y} = \frac{1}{2\pi}\vec{n}\cdot\nabla_y \ln\frac{1}{|x-y|}$

$\frac{\partial}{\partial y_1}\ln\frac{1}{|x-y|} = -\frac{1}{2}\frac{\partial}{\partial y_1}\ln|x-y|^2 = -\frac{1}{2}\frac{1}{|x-y|^2}\underbrace{\frac{\partial}{\partial y_1}[(x_1-y_1)^2+(x_2-y_2)^2]}_{-2(x_1-y_1)} = \frac{x_1-y_1}{|x-y|^2}$

so $\frac{\partial\Phi}{\partial n_y} = \frac{1}{2\pi}\frac{\vec{n}\cdot(\vec{x}-\vec{y})}{|x-y|^2}$

is harmonic for $x \neq y$.