1 Introduction

Numerical mathematics is at the intersection of analysis (devising and proving theorems), computation (devising algorithms, coding efficiently), and addressing application areas (e.g. PDE problems in engineering, science, technology).

This course will focus on the first two: analysis, and coding/testing computer algorithms. What is numerical analysis? Trefethen [1] gives an inspiring answer: it is not merely the study of rounding errors in computations, rather, it is the study of algorithms for the problems of continuous mathematics. We should also remind ourselves that carelessness over rounding errors, and over convergence issues, in numerical algorithms has caused loss of life and equipment destruction with losses of $10^8$ (see Arnold disasters website). Our goal is to understand the mathematics behind our algorithms, and be able to code them reliably and invent new ones.

Our topic is the solution of PDEs via integral equations (IEs). Along the way we touch upon rounding error, quadrature, numerical linear algebra, convergence, etc.

**Paradigm PDE:** Let $\Omega \subset \mathbb{R}^2$ be an open connected domain. (All of this works in higher dimensions too.) The interior BVP for Laplace’s equation is

\begin{align*}
\Delta u &= 0 \text{ in } \Omega \quad (1) \\
u &= f \text{ on } \partial \Omega \quad (2)
\end{align*}

where $\partial \Omega$ denotes the boundary of the set $\Omega$, i.e. the set of points that are both limit points of sequences in $\Omega$ and in $\mathbb{R}^2$.

**Omega.** The ‘boundary data’ is the given function $f$ on $\partial \Omega$. Applications include electrostatics ($u$ represents electric potential), steady-state heat distribution ($u$ is temperature), complex analysis ($u$ is the real part of an analytic function), and Brownian motion or diffusion ($u$ is probability density).

**Paradigm IE:** Let $[0, 1]$ be an interval, and we are given $f \in C([0, 1])$, and $k \in C([0, 1]^2)$ i.e. a continuous function on the unit square. Then find a function $u$ satisfying the integral equation

\[ u(t) + \int_0^1 k(t, s)u(s)ds = f(t) \quad \text{for all } t \in (0, 1) \quad (3) \]
This is a Fredholm equation, and since $u$ itself is present on the LHS, is called ‘2nd kind’.

To give an idea of the intimate connection between the above BVP and IE, consider that uniqueness for the BVP is easy to prove: Let $u$ and $v$ be solutions, then $w = u - v$ satisfies $\Delta w = 0$ in $\Omega$, and $w = 0$ on $\partial \Omega$. But by the maximum principle, the maximum of $w$ over $\Omega$ cannot exceed the maximum on $\partial \Omega$, which is zero. The same holds for $-w$, so $w \equiv 0$, and we have uniqueness. In contrast, existence of a solution to the BVP is much harder. It was first proved by transformation of the BVP to an IE, in 1900 by Fredholm, and, along with Hilbert’s work that decade, became the foundation of modern functional analysis. Here the identification is made between the 1D sets $\partial \Omega$ and $[0, 1]$. Thus the IE becomes a boundary integral equation or BIE.

The beautiful thing is that this method of proof leads to an efficient numerical method for solving the BVP. Crudely speaking, the efficiency stems from the reduction in dimensionality from $u$ being an unknown function in 2D in the BVP to only in 1D in the IE.

Waves: As well as Laplace, we will also study the Helmholtz equation

$$(\Delta + \omega^2)u = 0$$

where $\omega > 0$ is a frequency. What do solutions of this look like? The 1D analog is the ODE $u'' + \omega^2 u = 0$ which has solutions such as $\sin \omega x$ or $e^{i\omega x}$ which oscillate with wavelength $2\pi/\omega$. Similar things happen in higher dimensions, except that waves may travel in all directions. See picture

Notice that Laplace and Helmholtz are both elliptic PDE since the signs of the 2nd derivatives are the same. The contrasts with the wave equation, $\tilde{u}_{xx} + \tilde{u}_{yy} - \tilde{u}_{tt} = 0$ for the time-dependent field $\tilde{u}(x, y, t)$, which could represent acoustic pressure, for example. The wave equation is hyperbolic since its has mixed signs of 2nd derivatives. The mnemonic is to convert derivatives to powers of the coordinate (this is actually called the ‘symbol’ of a differential operator; see pseudodifferential operators):

\[
\begin{align*}
u_{xx} + u_{yy} = 0 & \iff x^2 + y^2 = \text{const} \iff \text{ellipse (here happens to be a circle)} \\
u_{xx} - u_{yy} = 0 & \iff x^2 - y^2 = \text{const} \iff \text{hyperbola}
\end{align*}
\]

Equations such as the heat equation have no 2nd-derivative in one of the variables, and are thus parabolic. Given even rough boundary data, elliptic PDEs lead to very smooth (even sometimes analytic) solutions; on the other hand, with hyperbolic PDEs rough initial data is carried along characteristics and remains nonsmooth. The picture for the wave equation is of the light cone disturbance produced by point-like initial data at the origin at $t = 0$.

The Helmholtz equation follows from the wave equation when the assumption of motion in time at a single frequency is made, e.g. if I were to sing in this
room with a pure tone at a single frequency, the pressure field would settle into one with ‘harmonic’ time-dependence

\[ \tilde{u}(x, y, t) = u(x, y)e^{-i\omega t} \]

Substitution of this into (5) and canceling exponential factors gives (4).

When waves traveling in free space hit an obstacle this is a scattering problem. One then needs to solve an exterior problem, with (4) holding in the unbounded domain \( \mathbb{R} \setminus \bar{\Omega} \), with given boundary data as before, and a so-called ‘radiation condition’.

What BIE methods are good for: Piecewise-homogeneous media, i.e. the coefficients of the PDE are constant in chunks of space touching on lower-dimensional boundaries. BIEs are excellent especially for exterior problems, finite element methods cannot easily handle the infinite extent of the domain. Also, BIE are excellent for high frequencies \( \omega \gg 1 \), since then there are many wavelengths across the domain, and the lower dimensionality of BIE vs FEM is a huge advantage.

What BIE methods are not good for: Variable-coefficient PDEs, or nonlinear PDEs. Note that there are IE methods for some of these, namely, volume-integral based methods such as Lippman-Schwinger.

2 Numerical Linear Algebra: Stability and Conditioning

Well, now we go over to scanned paper lectures…

(One day I will \TeX{} up the whole thing)

References

Numerical methods:
- Algorithms
- Theory
- Applications

Focus on these two:
- Numerical analysis
- Numerical methods

Topic: Solution of PDEs via integral equations (IEs)

- Given domain $\Omega \subset \mathbb{R}^n$; function $f$ on $\Omega$
- Find function $u(x)$ with $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ in $\Omega$.
- Seek $u(x) = \int_\Omega K(x,y) f(y) dy$ in $\mathbb{R}^n$.

Application: Electrodynamics, heat transfer, elasticity, wave propagation.

We will develop some PDE theory along the way.

Example of BVP is simple. But existence is hard: first prove existence. IEs (1880s-1910s: Fredholm, Hilbert).

The link between BVPs & IEs: Instead of $u$, use the boundary $\partial \Omega$.

We will need other key numerical areas such as numerical linear algebra, quadrature, in algebra, convergence...

30 min. Syll. (show online).
- HW: Unit 1: DFT probs, due Fri. (for the first few).
- Goal: understand numerical algos, code your own.
- X-hrs: Wed. & 3-3:50 for MATLAB exercises (please install MATLAB on your computer).
- Books: For a start. Take notes.

Waves: Different PDE: $(\Delta + \omega^2) u = 0$ Helmholtz, acoustics, also Maxwell in certain cases.

$- \omega = \text{freq. of wave}$

Solve oscillate in space. Why? 1D version: $u'' + \omega^2 u = 0$

eg solve for on website top.

Comes from wave eqn. for $u(x,y,t)$.

Assume const. freq. $u(x,y,t) = U(x,y) e^{-i \omega t}$ sub into $\Delta u + \omega^2 u = 0$ where $\omega = \text{wave speed}$.

Waves. 3D Helmholtz.

With homogeneous PDE, $\Delta u = 0$. Non-homogeneous $\Delta u = f$.

Boundary conditions.

A scattering: send in a solution to Helmholtz in $\mathbb{R}^3$ (eg plane wave), but which hits obstacle $\Omega$. Scatter.
solve exterior BVP
\[
\begin{cases}
\Delta u + w^2 u = 0 & \text{in } \mathbb{R}^2 \backslash \overline{\Omega} \\
\partial u = f & \text{on } \partial \Omega \\
\nu \cdot \nabla u = 0 & \text{at } \partial \Omega
\end{cases}
\]

who's seen?
\(\Omega = \) open set
\(\overline{\Omega} = \) closure.
so \(\mathbb{R}^2 \backslash \overline{\Omega} \) excludes \(\partial \Omega\).

50 mins. break?

(Trefethen book)

solve \(Ax = b\)
\(A \in \mathbb{C}^{m \times n}\)
\(x \in \mathbb{C}^n, \) matrix:
\[
\begin{bmatrix}
A^T & b
\end{bmatrix}^T \\
\begin{bmatrix}
A & b
\end{bmatrix}
\]

\(AX = b\)
\(x\) = \(Ax\)
\(x = A^{-1}b\)

Rescaling columns: what must \(A\) by if want \(col \ a_j\) to become \(dj\) \(\vec{d}\) where \(\vec{x} \in \mathbb{R}^m\) given?

\[
A \xrightarrow{D} \begin{bmatrix}
A & b
\end{bmatrix}
\]

postmultiply by \(D\).

what does \(D\) do? rescale rows.

\(\text{Space } \text{Col}A = \text{Span}\{a_j\} \subset \mathbb{R}^m\)

\(\text{Null } A = \text{all vecs with } Ax = 0 \subset \mathbb{R}^n\)

\(\text{dim } \text{Col } A \) called ? rank \(A\) = # points. \(\leq \min(m,n)\)

Say \(A\) full rank: when is \(A^\top A\) non-singular?

each \(y_j = Ax_j\) must be \underline{unique} (in combo if \(\{a_j\} \Rightarrow \text{cols (indep)} \Rightarrow \dim \text{Col } A = n\)).

\(\Rightarrow \text{Thm } (\text{full rank})\): \(A\) full rank \(\Leftrightarrow\) map 1-1 (solv to \(Ax = b\) unique, if exists).

Square case \(m \geq n\).

Full rank \(\Leftrightarrow\) \(A^\top A\) exists, \(A^\top A = A \cdot A^\top = I\). \(\Leftrightarrow\) soln \(x = A^{-1}b\) exists

unique.

App: polynomial approx.
\(\{x_j, y_j\} \) distinct set of reals
\(\text{Claim: mat } A \cdot \text{ element } a_{ij} = x_i^{j-1}\)
\(\Rightarrow \) \(x_1 \cdots n\) is non-singular.

How a arise? data \((x_j, y_j)\) \(j = 1, \ldots, p\) in plane.

What is \(n-1\) degree poly passing \(\text{through data}\)?

\(\vec{c} = \text{coeffs, } \vec{p}(x) = c_0 + c_1 x + c_2 x_2^2 + \cdots + c_{n-1} x^{n-1}\)
Let \( \mathbf{p} = \{ p(x_j) = y_j \} \) be the set of equations:

\[
\begin{cases}
  C_0 x_1 + \cdots + C_{n-1} x_1^{n-1} = y_1 \\
  C_0 x_n + \cdots + C_{n-1} x_n^{n-1} = y_n
\end{cases}
\]

where \( C_0, \ldots, C_{n-1} \) are constants.

Suppose \( \mathbf{c} \neq \mathbf{c}' \) are two such solutions.

Then, \( p(x) - p'(x) \) is a nonzero, degree-\((n-1)\) polynomial vanishing at each \( x_j \), i.e., have \( n \) distinct roots. This is impossible. So \( \mathbf{c} \) is unique if \( \mathbf{A} \) is full rank. \( \square \)

Notice that:

\[
\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}, \quad \text{rank}(\mathbf{A}) = 2.
\]

Always \( \frac{1}{2} < \frac{1}{3} < \frac{1}{4} \) and \( \mathbf{A} \) is nonsingular and consists of columns.

\[
\mathbf{A} \quad \text{hard to see this by eye} \rightarrow \text{ways to view (image of)}
\]

\[
\text{rank}(\mathbf{A}) = \frac{1}{3} \quad \text{as usual says.}
\]

Try \( n = 30, 40, 60 : \quad \text{rank}(\mathbf{A}) \text{ norm } > 36 \quad \text{why?}
\]

\( n = 10 \) size of \( \mathbf{A} \)
\( n = 20 \) size of \( \mathbf{A} \)

\[
\mathbf{10}^4 \quad \text{ grows!} \quad \mathbf{40} \quad \text{ needs singular value of } \mathbf{A}
\]

\[
\mathbf{10}^6 \quad \text{when size } 40 \quad \text{someplace!}
\]
Orthogonality (review in alg.)
A matrix, $A^*$ hermitian transpose. \( (A^*)_{ij} = \bar{A}_{ji} \) c.c.
\[ A = A^* \text{ symm.} \]

Ex. prove \( (AB)^* = B^*A^* \), \( (A^{-1})^* = (A^*)^{-1} \)
\[ \times \text{ col. vec.} \times^* \text{ row vec.} \]
\[ x, y \in \mathbb{C}^n; x \cdot y = \? \text{ inner prod } \mathbb{C}^n \]
\[ \langle x | y \rangle = \sum_{i=1}^{n} \bar{x}_i y_i \]
2-norm: \( \|x\|_2 = \sqrt{x^*x} \)

Defn: 4 a norm?
\[ \begin{align*}
& i) \|cx\| = |c| \|x\| \quad \text{a scalar} \\
& ii) \|x\| = 0 \Rightarrow x = 0 \quad \text{a vector} \\
& iii) \|x + y\| \leq \|x\| + \|y\| \quad \text{triangle} \\
\end{align*} \]

2-norm also has \( \|xy\| \leq \|x\| \|y\| \) Cauchy-Schwarz
\[ x^*y = 0 \? \quad x, y \text{ orthog.} \]

Then: \( \text{ mutually orthog. set of vectors are lin. indep. \ (pf: Ex)} \)
\( \Rightarrow \text{ orthog. rows in } \mathbb{C}^n \) from basis; \( \text{ if unit length, \ o.n.b.} \)

Say \( \vec{x}_1, \ldots, \vec{x}_n \in \mathbb{C}^n \) o.n.b., can stack into cols. of \( Q \), then \( (Q^*Q)_{ij} = \sum_j Q_{ij} \bar{Q}_{ji} \)
\[ \begin{align*}
\text{so } Q^*Q &= I \\
\Rightarrow Q^{-1} &= Q^* \quad \text{cols o.n.b. \leftrightarrow unitary} \\
\end{align*} \]

so \( Q^*b \) is coeff of expansion of \( b \) in o.n.b. \( \vec{x}_i \)
\[ \|Qx\| = \sqrt{(Q^*Q)x^*x} = \sqrt{x^*Q^*Qx} = \|x\| \quad \text{preserves length: \ (rotation \ poss w/shift)} \]

Matrix 2-norm: \( \|A\|_2 \) smaller \( \iff \|Ax\| \leq \|A\|\|x\| \quad \forall x \in \mathbb{C}^n \)
\[ \text{implies } \|A\|_2 \leq \sigma_{\text{max}} A \]

or \( \|A\| : = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| \)

Eg: \( \|A\| \) of diag matrix? Its largest-magnitude entry.

ii) 2-norm of \( A = uv^* \) \( (\text{rank-1}) \): \( \|uv^*x\|_2 \leq \|v^*x\|_2 \cdot \|x\|_2 \leq \|v\|_2 \|x\|_2 \leq \|u\|_2 \|x\|_2 \leq \|v^*x\|_2 \leq \|u\|_2 \|v\|_2 \|x\|_2 \)

2-norm submultiplicative: \( \|AB\| \leq \|A\| \|B\| \)

Ex. show \( QA \) & \( A^*Q \) have same 2-norm as \( A \). \ (Thm 3.1)
Singular Value Decomposition (SVD) — is important as spectral decomp.

Geometric fact: every $A \in \mathbb{C}^{m \times n}$ maps unit ball into hyperellipsoid.

Full case $m \geq n$ & full rank: $A = U \Sigma V^*$.

Left sing. vecs. $u_1, \ldots, u_n$ orthonormal.

Sing. vals. $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$.

Semi-axes.

Right sing. vecs. $v_1, \ldots, v_n$ unit.

Premultiplication of $v_j$ on right.

$\sigma_1 \geq \cdots \geq \sigma_n > 0$ while $\sigma_{n+1} = \cdots = \sigma_n = 0$.

So $A v_j = \sigma_j u_j$.

$A = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$.

Reduced SVD.

Can complete $U$ to $U_{mn}$ or $V$ for $V^*$.

Then $A = U \Sigma V^*$.

Every matrix is rotation $\rightarrow$ stretch $\rightarrow$ (or refl.) even nonsym or nonsquare.

or: every matrix is diag in correct basis for $\mathbb{C}^n \otimes \mathbb{C}^m$ ($V^*$ projects to cols. of $A$ only).

If $A$ square invertible, $A^{-1} = (U \Sigma V^*)^{-1} = V \Sigma^{-1} U^*$.

Is the SVD of $A^{-1}$ (if exists).

Diag $\sigma_j^{-1}$.

What is $\|A^{-1}\|$? Largest sing. value.

$\sigma_n^{-1}$ smallest sing. of $A$.

Needed for $\text{WS}$. (PTO).

Notes on SVD: existence of either by induction (NLA Ch. 4 — beautiful prof. grad. students read).


or, $A^* A$ has eigenvalues $\sigma_j^2$ & complete eigenvectors $v_j$. (why?)

$A A^*$.

Eigenvectors $\sigma_j \geq 0$ & then define $\sigma_j = \sqrt{\sigma_j}$.
Anatomy: SVD & spaces:

\[
\begin{bmatrix}
A
\end{bmatrix} = 
\begin{bmatrix}
U
\end{bmatrix} \Sigma \begin{bmatrix}
V^T
\end{bmatrix}
\]

\( \text{Rank} \): \( \text{rank} \, A \)

Null space: \( \text{null}(A) \)

Column space: \( \text{col}(A) \)

Null space of \( A^T \): \( \text{null}(A^T) \)

Null space of \( A \): \( \text{null}(A) \)

Null space of \( A^T \): \( \text{null}(A^T) \)

Numerical rank: \( r_\text{num} \)

Machine precision: \( \varepsilon \approx 10^{-16} \)

Conditioning: \( (\text{SVD NLA}) \)

Problem is well-conditioned if

\[ \text{Problem is well-conditioned if } \lim_{x \to 0^+} \frac{\|f(x + \epsilon) - f(x)\|}{\|\epsilon\|} = 0 \]

Condition number:

\[ K = \frac{\sup_{x \neq 0} \frac{\|f(x + \epsilon) - f(x)\|}{\|\epsilon\|}}{\sup_{x \neq 0} \frac{\|f(x)\|}{\|x\|}} \]

If \( x, f \) vectors, \( \frac{df}{dx} = J_f(x) \) is Jacobian matrix.

Basic ops:

\[ f(x) = x^2 \quad \text{(near)} \quad J_f = 2x \]

\[ f(x) = \frac{x}{x} \quad \text{subtraction} \quad J_f = 0 \]

\[ f(x) = \sin(x) \quad \text{near} \quad J_f = \cos(x) \]

\[ f(x) = x^{100} \quad \text{for} \quad x = 10^{100} \quad \text{say:} \quad \|f\| \leq 1 \quad \text{but} \quad K = \frac{\|f\|}{\|x\|} \quad \text{large} \]

Finding poly roots, ill-conditioned.

\[ \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \]

Ill conditioned: \( K \approx 10^{10} \)

\( K \approx 1 \)

But symmetric, \( K \approx 1 \).
rank \( r := \# \{ j : \sigma_j > 0 \} \)

numerical rank \( r_e := \# \{ j : \sigma_j > \varepsilon \} \) in \( \mathbb{C} \) (machine precision)

conditioning (SVD NLA): property of a well problem (vs. stability: property of alg used to solve \( Ax = b \))

problem is map \( f : X \rightarrow Y \) space of

\[ f \text{ well-const if infinitesimal pert } \delta x \text{ a } x \text{ cause small pert } \delta y \text{ to } y \]

\[ \| \delta y \| \leq \| \delta x \| \text{ as } \delta x \rightarrow 0 \]

Abs. cond (condition number): \( \kappa = \kappa_f(x) := \lim_{\| \delta x \| \rightarrow 0} \sup_{\| \delta x \| \leq \varepsilon} \frac{\| f(\delta x) \|}{\| \delta x \|} \)

Rel. cond (relative condition number): \( \kappa = \kappa_f(x) := \frac{\| f(x) \|}{\| f(x) \|} \frac{\| \delta x \|}{\| \delta x \|} \)

if \( x, f \) vectors, \( \frac{df}{dx} = J \) is Jacobian, matrix \( J \in \mathbb{C}^{m \times n} \)

As \( \| \delta x \| \rightarrow 0 \) have \( \delta f \approx J(\delta x) \) so \( \kappa_f(x) = \frac{\| J(x) \|}{\| J(x) \|} \)

Basic ops:
- \( f(x) = x/2 \) (mean): \( J = f'(x) = 1/2 \) so \( \kappa = 1/2 \)
- \( f(x) = x^x \) \( J = f'(x) = x \times x^{x-1} \) \( \kappa = 1/x \times x^{x-1} \times x^{x-1} \) \( \approx 1 \times 1 \times 1 \) \( \approx 1 \) well-cond for reasonable phases,
- \( f(x) = x_1 - x_2 \) subtraction (m=2, n=1): \( J = [1 \ 1] \) \( \| J \| = \sqrt{2} \)
  \( \kappa = \frac{\sqrt{2} \sqrt{x_1 - x_2}}{x_1 - x_2} \) as \( x_1, x_2 \neq 0 \) can be ill-cond.
- \( f(x) = \sin x \) for \( x \ll 10^{10} \) say: \( \| J \| \leq 1 \) but \( \kappa = \frac{\| J \|}{\| \delta x \|} \ll 1 \) large.
  (Last abscissa\# \( \leq 1 \))
- \( f(x) = x \) poly root, ill-cond
- \( \gamma \) almost nonsymmetrics: eg \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) vs \( A = \begin{bmatrix} 1 & 10^{-3} \\ 10^{-3} & 1 \end{bmatrix} \)
  \( \| A \| = 1 \) \( \| \delta x \| = 10^{-3} \) \( \| \delta x \| = 1 \) \( \kappa \approx 10^3 \), \( \kappa \approx 1 \)

but symm, \( \kappa = 1 \).
Natural w.m.d. \( f(x) = Ax \) \( \xrightarrow{\text{J:\text{2A}}} K \) \( \|A\| \) (\( \text{def. 2-norm} \)) \( \Rightarrow \) if \( A \) nonsing., \( \|K\| \leq \|A\| \cdot \|A^{-1}\| \), why?

so \( K \leq \|A\| \cdot \|A^{-1}\| \)

Solving lin sys? \( Ax = b \) so \( f(b) = A^{-1}b \)

replace \( A \) by \( A^{-1} \) as above, get again \( K \leq \|A\|^{-1} \cdot \|A\| \)

with equality if \( x = v_m \), def of 2-norm. \( \Box \)

so, tight. \( \Box \)

Ex: check.

call \( \|A\|\|A^{-1}\| = \kappa(A) \) cond of matrix \( A \) =

\[ \frac{\|A\|}{\|A^{-1}\|} = \text{ eccentricity of hyperellipse.} \]

Ex: check.

What if \( A \) perturbed instead, solving lin sys? \( ^2 \) input is \( A \), output \( x \) (\( b \) held const).

\[
\text{consider infinitesimal changes:}
\begin{align*}
(A + \delta A)(x + \delta x) &= b \\
\text{so} \quad \delta A \text{ causes} \quad \delta x
\end{align*}
\]

solving \( \|\delta A\| \cdot \|\delta x\| \leq \|A\| \cdot \|\delta x\| \)

so \( \delta x = -A^{-1} \delta A x \)

rel cond of

\[
\frac{\|\delta x\|}{\|x\|} \leq \|A\|^{-1} \cdot \|A\| = \kappa(A) \text{ again.}
\]

\( \Rightarrow \) since \( A^{-1}b \) stored to 16 digits, expect to get \( x \) to 16 - \log_{10} \kappa \) digits acc.

Floating Point

\( \mathbb{R} \): digital rep. : finite # bits \( \Rightarrow \) finite subset \( F \) of \( \mathbb{R} \) |mostly 2s (16384) |

\( [\pm 2^{-3}, \ldots , 2] \)

eg \( [1,2] \) rep by set \( \{ 1, 1 + 2^{-52}, 1 + 2.2^{-52}, \ldots , 2 \} \)

\( [2,4] \) is twice these \( \text{(larger gaps)} \) rel. gap \( 2.2 \times 10^{-16} \) never exceeded

but a poor algorithm can cause this to dominate.

Formally, base \( \beta = 2 \), precision \( \tau = 53 \)

set \( F = \{ 0, \pm \frac{m}{\beta^t}, \pm \text{Inf}, \text{NaN} \} \)

\[
\text{special code, rather than members of } \mathbb{R}.
\]

\( \beta^{-1} \leq m \leq \beta^t \)

so \( \frac{m}{\beta^t} \in [-1,1] \)

Note \( \beta F = F \) self-similar.

Ex: \( x \): \( \|x\| \cdot \|x\|^{-1} = 1 \) is largest rel. gap.

\( \|x\|^{-1} = \|x\| \cdot \|x\|^{-1} \) is largest rel. error.

\( \Rightarrow \) let such \( x \) be called \( f_1(x) \)

Then \( \forall \mathbb{R}, \exists \epsilon, 1 \leq \|x\| \leq E_{\text{mach}}, \exists \|x\|^{-1} = \text{mach} \|x\|^{-1} \)

\( \Rightarrow \) let such \( x \) be called \( f_1(x) \)

IEEE double precision \( E_{\text{mach}} = 2^{-53} \approx 1.1 \times 10^{-16} \)
Arithmetic: let \(\oplus\) be any of: could require \(x \ominus y = f_1(x \ominus y)\), ie gives the unique rounding of answer.

But only need weaker: Fund. Axiom of Floating Pt:

\[\forall x, y \in F \exists \lim_{\varepsilon \to 0} \text{ st. } x \oplus y = (1 + \varepsilon)(x \ominus y)\]

if \(\varepsilon\) is built by machine.

For \(C\) instead of \(R\), turns out to be \(2^{32}\) or \(2^{64}\), similar.

Stability (§14 MLA) alg getting stuck was even if not exact.

For:\\\n- problem \(f: X \to Y\) \(\text{eg. } y = \sin x\) or \(y\) is sol to \(A y = b\) (here \(x, y, a, b\) is \(A, b\))
- \(f, \hat{f}\): alg
- algorithm for \(f\) also \(\hat{f}\): alg

Steps in alg: \(\rightarrow\) \(\rightarrow\) \(\rightarrow\) \(\rightarrow\)

\[\frac{\|\hat{f}(x) - f(x)\|}{\|f(x)\|} \leq \text{ alg. certain good if this is } O(\varepsilon\text{mach})\]

- of the order of 1

Formally: \(O(\varepsilon\text{mach})\) means: \(\varepsilon\text{mach}\) is a family of floating pt sys, for some const \(C\), uniformly over all data \(x \in X\).

Practically: \(<10^{-2}\varepsilon\text{mach} \text{ok}, >10^{8}\varepsilon\text{mach} \text{ not ok}\).

But if problem \(f\) ill-cond, unreasonable to demand this! Why? rounding on input changes \(x \rightarrow f(x)\) \(\varepsilon\) if \(x\) very big, change gets blown up by \(K\), so even if alg exact, must have \(O(\varepsilon\text{mach})\) relative in output.

Instead: defn

Alg stable if \(\forall x \in X\), \(\frac{\|\hat{f}(x) - f(x)\|}{\|f(x)\|} = O(\varepsilon\text{mach})\)

for some \(\varepsilon\) s.t. \(\frac{\|x - \hat{x}\|}{\|x\|} = O(\varepsilon\text{mach})\)

"nearly right answer to nearly right question."

Stronger: Backward stable: \(\forall x \in X\), \(\hat{f}(x) = f(x)\) for some \(\varepsilon\) s.t. \(\frac{\|x - \hat{x}\|}{\|x\|} = O(\varepsilon\text{mach})\)

"exactly right ans. to nearly right question."

eg: is \(\Theta\) blow stable? \(\text{par : } f(x_1, \ldots, x_n) = x_1 - x_i\)

alg: \(\hat{f}(x_1, \ldots, x_n) = f(x_1) \ominus \cdots \ominus f(x_n)\)

\[= (x_1 (1 - \varepsilon)) - x_i (1 + \varepsilon)\text{ s.t. } x_i (1 + \varepsilon) - x_i (1 + \varepsilon) = f(x_i, x_i)\text{ exact for some data}\]

for \(\varepsilon < \varepsilon\text{mach} \leq 1/43\).
Is \( f(x) = x \Theta 1 \) blow st? \( f(x) = [x(1+E_2) - 1] (1+E_2) = x(1+E_2) - 1 = f(x) \) \( \Box \)

how b) \( E_3 \)? \( xE_3 = x(\varepsilon_1 + \varepsilon_2 + O(\varepsilon^3)) - E_2 \)
so \( \varepsilon_3 = \varepsilon_1 + \varepsilon_2 - \frac{E_2}{x} = O(\varepsilon_1 + \varepsilon_2) \)
So as \( x \to 0 \), not blow st. But, \( \varepsilon \) stable.

Some algs unst! (e.g. polyroots)

Take-home msg: algs (in Matlab, DMPACK, etc) for solving \( Ay = b \) are blow stable (\( A \) is dense, \( y \) is augm)

\( m \geq n \): least-squares soln, i.e. find \( x \) st: \( \|Ax-b\| \) minimized.

is blow stable w/ SVD (Thm 19.4) How do? \( \tilde{A}b \) w/ divide 
\& or, explicitly, \( A = \hat{U} \Sigma \hat{V}^T \) so \( x = \hat{V}^T \Sigma^{\frac{-1}{2}} \tilde{U}b = : A^+ b \)

Reminder: \( \frac{\|\hat{y} - y\|}{\|y\|} \) may not be small, i.e. \( y \) itself inaccurate! What is only cause of this m blow-stable alg? \( K \) v. large.

But in this case, such is blow st; as good as one could hope for!

How accurate is \( \hat{y} \)? For any blow-stable alg: \( x \to f(x) \) Thm (15.1): if cond \( \{x \} \) for prob \( f(x) \), alg is blow st, and compute obeys floating pt axing, then rel. err. satisfies
\[ \frac{\|f(x) - f(\tilde{x})\|}{\|f(x)\|} = O(K(x) \varepsilon_{mach}) \]

i.e. error is \( K \) times worse. If \( K > 10^5 \), you lose all digits of \( f(x) \), or \( \tilde{y} \). But it still holds that

\( \tilde{y} = b \) exactly!
\( A^+ b = b \)

Proofs by defn. \( f(x) = f(x) \) (a) for some \( \|x - y\| \leq O(\varepsilon_{mach}) \) (b) since not infinitesimal.

Stability: finally done.
Interpolation \[ \text{Approximate } f \text{ on } \mathbb{R}^n \text{ by degree-}n \text{ poly } p(x) = \sum_{k=0}^{n} p_k x^k \]

If choose not distinct \( x_i \) to make \( f \) & \( p \) match, we're already solved this:
\[
\begin{align*}
p(x_j) &= y_j, & j = 0, \ldots, n \quad \text{so} \quad \begin{bmatrix} y_0 & x_0^1 & \cdots & x_0^n \\ y_1 & x_1^1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ y_n & x_n^1 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix} \quad \text{pencil det } M \neq 0 \implies \text{set } a \in \mathbb{R} \text{ exists, unique.}
\end{align*}
\]

Let \( L_k(x) = \prod_{j \neq k} \frac{x-x_j}{x_k-x_j}, \quad k = 0, \ldots, n \) \text{ called Lagrange basis (1772)}

Prop: \text{unique poly } p = \sum_{k=0}^{n} y_k L_k \quad \text{why? } L_k(x_i) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}
\]

\[ p(x_i) = \sum_{k=0}^{n} y_k L_k(x_i) = y_i \quad \text{via since one factor } (x_i-x_i) \]

Note: \( n \) large \((>30)\) may cause stability prob since \( \max \lvert L_k(x) \rvert \exp \text{ large} \)

- Newton 1676 realized more practical method (divided differences) \( \rightarrow \) we don't need.

- \text{map } f \rightarrow \text{its unique interp poly } p \text{ through } \{x_j\} \text{ is linear: } p = \text{Ln} f \text{ for any } p \in P_n, \quad L_n p = p \quad \text{so what kind of op is } \text{Ln} ? \text{ projection: } L_n^2 = L_n.

Error of interp. \( \text{Ln} f - f \) is a func.

- Recall \( C^k[a,b] \) space of \( k \)-times cont diff. funcns, \( \text{ie } k^{th} \text{ der. is cont.} \)

Thm 8.1: Let \( f \in C^{n+1}[a,b], \text{ then for each } x \in [a,b] \text{ there exists } \xi \in [a,b] \text{ st. } \)
\[
f(x) - \text{Ln} f(x) = \frac{1}{n!} \sum_{j=0}^{n} \frac{f^{(n+1)}(\xi)(x-x_j)^{n+1}}{n!(x-x_j)}
\]

- So if you knew \( \lvert f^{(n+1)}(x) \rvert \leq C \text{ in } [a,b] \text{ you get on error estimate } a \text{ mathemat. rigorous bound!}

If \( f \) \text{ trivial if } x = x_j

Fix \( x = x_j \) & define \( g(x) := f(x) - \text{Ln} f(x) = \sum_{j=0}^{n} \frac{f(x) - \text{Ln} f(x)}{n!(x-x_j)^{n+1}} 
\)

\[
\begin{align*}
g(y) &= \frac{\sum_{j=0}^{n} (y-x_j)^{n+1}}{n!(x-x_j)} \quad y \in [a,b] \\
y &= x_j & g(x_j) = 0 \text{ too! (that was only constructed, so } g\text{ has no } \text{zeros in } [a,b] \\
\text{By Rolle's Thm, } &g^{(n)} \text{ has } \geq 1 \text{ zeros, call it } \xi \\
\text{etc.} &g^{(n+1)} \text{ has } > a \text{ zeros, call it } \eta \\
\end{align*}
\]

Set \( y = \xi \) & eval \( f^{(n+1)}(\xi) : 0 = f^{(n+1)}(\xi) - 0 - \text{(n+1)!} \frac{f(x) - \text{Ln} f(x)}{n!(x-x_j)} \quad \text{for sure!} \)
Write top of VS on board.

US on n=1 lagrange.

Some apples.

deviation - potential, etc.

\[ f(x) = \frac{1}{1 - 25x^2} \]

\[ x_n = -1 + \frac{3i}{n} \quad \text{in } E \] ( laurent applet)

interpolant \( L_n \) \( \text{blows up } \infty \) in outer regions of \( E \). Why?

\[ \text{Why?} \]

If assume nothing about nodes, product \( \prod (x-x_n) \) \( \leq (b-a)^n \Rightarrow \text{uniform bound}. \)

Thus \( \| f-L_n \|_\infty \leq \frac{M^n}{n!} \) \( \text{length of interval}. \)

We want small!

How big 

Recall Taylor series \( f(x) \approx a_n x^n \quad \text{about } a \)

abs. \( a_n \to 0 \quad \text{as } n \to \infty \)

\( \Rightarrow |a_n(p-a)^n| \to 0 \quad \text{for } |p-a| < \beta \)

\( \Rightarrow f \) analytic in \( |z| < \beta \).

Lest say \( f \) analytic in open domain containing \( S \), then Taylor series converges in disc \( |z| < \rho \)

so if \( f \) analytic in \( |z| < \rho \) get good convergence regardless of nodes.

But if \( f \) analytic in neighborhood of \( \{a_i\} \), but singularities are \( \beta \) or closer, may fail to converge.

I leave for you to say where pole of \( \frac{1}{1 - 25x^2} \) are! (Remy)

The bad news: if construct seq. \( \{L_n\} \) each with \( \{x_n\} \), no nodes,

Then (8.17) \( \text{Fabry): for each such seq. } f \in C[a,b] \text{ st. } L_n \to f \text{ uniform on } [a,b] \).

The good news: Thm (8-16, Malgrange) For each \( f \in C[a,b] \), \( \exists \) seq. of nodes \( \{a_i\} \) \( \text{st. } L_n \to f \text{ uniform on } [a,b] \).
Why keek to cluster nodes at ends of L [?](Trefethen, Sec. 8.1)?

\[-\ln \left| \prod_{j=0}^{n} |z-x_j| \right| = -\ln \sum_{j=0}^{n} \frac{1}{|z-x_j|} = \text{electrostatic potential in } \mathbb{R}^2 = \mathbb{C} \]

due to net charge strength \(-\frac{1}{n!}\) at nodes.

\[-\ln |q_{n+1}(z)| = e^{-(n+1)\phi(z)}\]

Say as n->, nodes tend to fixed density finite \(\rho(x)>0\) on [1,1], then \(\phi_{n+1} \to -\sum x |\rho(x)| dx \), \(\phi(1) = -\sum x |\rho(x)| dx\).

Uniform case \(\rho=\frac{1}{2}\) so \(\phi(z) = \frac{1}{2} \int_1^z \ln |z-x| dx \rightarrow \text{you can eval.}\)

\[\text{& check } \phi(0) = -1 \quad \text{but } \phi(1) = -1 + \ln 2 \quad \text{larger at ends}.\]

\[\text{so } |q_{n+1}| \approx e^{(n+1)\ln 2} = 2^{n+1} \quad \text{grow bigger at ends}.\]

\[\text{Show change: potential. in.}\]

This same applies to \(\ln(x): \text{for } x/2 \text{ in middle, exp. large at ends } \rightarrow \text{unstable.}\]

Given smallest pass have \(\phi(2) \text{ hence smallest } |q_{n+1}|), \text{book interp. converges.}\]

\[\text{can show singularities of } f \text{ can be evb close to } 0/6 \text{ & still get exponential conv.}\]

\(\text{L: spectral method).}\]

9.1 Quadrature.

\[\text{want to approximate } \phi(f) := \int_0^1 f(x) dx \]

\[\phi_n(f) := \sum_{k=0}^{n} w_k f(kh) \text{ weights at node in } [0,1].\]

Given node, what are good weights? Pick st. \(\phi_n(f) = \sum_{k=0}^{n} \Delta_k f(kh) \text{ is integrate the interpolating poly exactly.}\]

\[\text{Then (9.2) give distinct node } \Delta_k \text{, the above.}\]

\[\text{Evaluating are the unique set which integrates all } p \in \mathbb{P}_n \text{ exactly.}\]

\[\text{since } f = \sum \Delta_k f(kh) \quad \text{exactly.}\]

\[\text{so integrator is exact.}\]

\[\text{So exact integration up to degree n can be taken as defining feature: called Newton-Cotes (sometimes only Gauss-\text{equidistant).}\]
\[ w_0 = \int_0^b \ell(x) \, dx = \int_0^b \frac{x - b}{x - b} \, dx = \frac{1}{2} (b - a) = \frac{\theta b}{2} \quad \text{if } \theta = \text{some} \]

so \[ Q_1(f) = \frac{1}{2} (f(a) + f(b) = \left\lfloor \int_a^b f(x) \right\rfloor \text{ truncated value.} \]

**Error and \( \theta \)?** Then 9.4: \( f \in C^2([a,b]) \), hence \( \int_a^b f(x) \, dx = Q_1(f) = -\frac{1}{2} f''(\xi) \) for some \( \xi \in [a,b] \)

**pf:** \( E_1(f) = \int_a^b (f(x) - L_1 f(x)) \, dx = \int_a^b \frac{(x-a)(x-b)}{2} \left( \frac{f(x) - L_1 f(x)}{x-a} \right) \, dx \)

\[ \leq 0 \quad \text{conv. by l'Hopital at endpoints.} \]

MVT for integrals: if \( g > 0, f \in C \), then \( \int_a^b g f \, dx = g(\xi) \int_a^b f \, dx \) for some \( \xi \in (a,b) \)

so \( E_1(f) = \frac{f(x) - L_1 f(x)}{(x-a)(x-b)} \int_a^b (x-a)(x-b) \, dx \)

by Thm 3.10 last time

\[ = \frac{f''(\xi)}{2} \quad \text{some } \xi. \]
Lec 6. MLE

last time: Interpolating quad on \(x_0, x_1, \ldots, x_n\), choose \(x_0 = a, x_n = b\), get \(Q_n(f) = \int_a^b f(x) \phi_n(x) dx\) i.e., exact \(\forall \phi_n\).

Thm 9.1: Let \(f \in C^1[a,b]\), then

\[
\left| \int_a^b f(x) dx - Q_n(f) \right| \leq \frac{b-a}{n+1} \int_a^b f(x) dx^2 = \frac{b-a}{n+1} \sup_{c \in [a,b]} |f'(c)| \int_a^b dx^2 = \frac{b-a}{n+1} \sup_{c \in [a,b]} |f'(c)|
\]

For \(n = 2, 3\),

\[
\left| \int_a^b f(x) dx - Q_n(f) \right| \leq \frac{b-a}{n+1} \sup_{c \in [a,b]} |f'(c)| \int_a^b dx^2 = \frac{b-a}{n+1} \sup_{c \in [a,b]} |f'(c)| (b-a)^2 = O(h^2)
\]

\(\Rightarrow\) local order.

Another way to obtain negative weight bad: Convergence.

(9.2)

Consider seq \(Q_n(f)_n\) or otherwise: \(Q_n(f) = \sum_{j=0}^{n} w_j f(x_j)\)

Define \((Q_n)\) conv. if \(Q_n(f) \rightarrow Q(f)\) as \(n \rightarrow \infty\), \(\forall f \in C^1[a,b]\) nice property! (Recall: impossible for integrals!)

Thm (Szegő): \((Q_n)\) conv. \(\iff\) \((Q_n)\) conv. for all polynomials \(P \in \mathbb{P}_k\), \(\forall f \in C^1[a,b]\), nice property! (Recall: impossible for integrals!)

note: these norms if weights thru up as \(n \rightarrow \infty\), cannot be cons! (e.g., equispaced)

Facts 1) \(R\) = poly.'s dense in \(C[a,b]\), ranging: \(\forall f \in C[a,b], \exists \phi = \sum_{j=0}^{k} \phi_j x^j, k \leq n\),

- if need densification: like \(\mathbb{R} < \mathbb{R}, \) but metric space is \(C[a,b]\) not sup norm.

2) each \(\phi_j\) is lin. op: \(C[a,b] \rightarrow R\) with \(\|Q_n(f)\| \leq \sup_{|x| \leq 1} |f(x)| \leq C\), sup norm.

DF: use facts 1, 2 w/ Banach-Steinhaus Thm:

in every case, \(\|f\| \leq C\), sup norm.

Let \((Q_n)\) be seq of bounded lin. ops, \(X = C[a,b]\) Banach space, \(Q_n(f)\) op, \(\{Q_n(f)\}\) a closed subset of \(X\).

Pointwise convergence \(\iff\) \((Q_n)\) uniformly bounded & convergent in norm.

\[
\forall f \in X, \lim_{n \rightarrow \infty} \|Q_n(f) - Q(f)\| = 0
\]

\(\Rightarrow\) \((Q_n)\) uniformly bounded & convergent in norm.

\(\iff\) \((Q_n)\) convergent in norm.

Therefore, \(\|Q_n\| \leq C, V_n\)

\(\lim_{n \rightarrow \infty} \|Q_n - Q\| = 0.\)
For any $\epsilon > 0$,

$$Q_{\text{unif}} - Q_{\text{unif}} = Q_{\text{unif}} - Q_{\text{unif}} + Q_{\text{unif}} - Q_{\text{unif}} - Q_{\text{unif}}$$

can be made $p$-\text{close} to $0$.

Take $\delta = \epsilon / \|Q_{\text{unif}}\|$ for some $N$ suff. large.

For each $\delta > 0$, choose $\epsilon = \frac{\delta}{\|Q_{\text{unif}}\|}$.

So, for each $\delta > 0$, choose $\epsilon = \frac{\delta}{\|Q_{\text{unif}}\|}$ and $N$ such that $\forall N > N$, $\|Q_{\text{unif}} - Q_{\text{unif}}\| < \delta$.

Positive weights is enough:

**Corollary (1.11, Steklov):** If $(u_n)$ conv for all polys $x^k w_{j,n}^{(m)} > 0$, then $(u_n)$ conv.

$$|Q_{\text{unif}}| = \sum_{j=0}^{\infty} |W_{j,n}^{(m)}| = \sum_{j=0}^{\infty} W_{j,n}^{(m)} = Q_{\text{unif}}(x) \xrightarrow{\text{pol}} \text{poly}$$

So there exists $M_{\text{unif}} = C$ in $Y$, use Szegö.

---

**Gaussian Quadrature (9.3)**: Optimal choice of nodes $\rightarrow$ quasi of quadratures on $[a,b]$.

G. is straight on.

Let $p$ of degree $S$ exist, generally $< \frac{2}{3}$ degree $S$ will not be $\text{exact}$ compared to only $n$ for Niederreiter/Lattice.

Let's show why:

- Orthogonality for Gausses: $f + g \Rightarrow D = \langle f,g \rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx = \sum_{n} \frac{w_{n}}{p_{n}}$.

**Lemma (9.13)** Let $x_0, \ldots, x_n$ be distinct nodes of a Gaussian quadrature.

Then $Q_{\text{unif}}(x) = \prod_{i<j}^{n+1} (x-x_j) \neq 0$ for each node.

**Proof:** $Q_{\text{unif}} \in P_{n+1}$ so $\sum_{x} Q_{\text{unif}}(x)p(x)\,dx = \int_{-\infty}^{\infty} (x-x_j) p(x)\,dx = 0 \forall x_j$.
Claim 2: $n+1$ is highest pass degree for $(n+1)$-mode quadrature.

Let $p = \frac{1}{n+1} \sum_{j=0}^{n} (x-j)^2 \in P_{n+1}$ has $Q_n(p) = 0$ but $Q(p) > 0$.

Thus (1.8) same weight non-negative. If $L_k(x) \cdot L_j \Rightarrow L_k^2(x) = 6k + 1$

\[ V_k = 0 < \int_{-\infty}^{\infty} L_k(x)^2 \, dx = \sum_{j=0}^{n} w_j L_j(x)^2 = w_k. \]

Cor: Gaussian quadrature (last term).

There are several forms for Gaussian quadrature. eg. order rule: 

\[ \text{error} \leq \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} |f^{(2n)}(x)| \, dx, \]

may generate more weighted product. $Q(p) = \int_{-\infty}^{\infty} f(x) W_k(x) \, dx$, has some uses.

**Periodic Quadrature:**

\[ f(x+2\pi) = f(x) V_k. \]

\[ Q_n(f) = \int_{0}^{2\pi} f(x) \, dx \]

\[ Q_n(f) = \frac{2\pi}{n} \sum_{j=0}^{n-1} f \left( \frac{2\pi j}{n} \right) \quad n \text{ equispaced nodes.} \]

Thus (9.2) Let $f \in C^{2n+1} (\mathbb{R})$ be $2\pi$-periodic, \( \omega_n \) \( \Rightarrow \) then \( |Q_n(f) - Q(f)| \leq C \int_{0}^{2\pi} |f^{(2n+1)}(x)| \, dx \)

If $f \in C^{\omega}$, then error $= O(n^{-\omega})$ for each $n \geq 0$, called super-algebraic convergence.

But if $f$ analytic, do even better; exponential conv.

\[ \int_{0}^{2\pi} f(x) \, dx = 2\pi i \sum_{\text{residues}} \text{residues}. \]

\[ \int_{0}^{2\pi} f(x) \, dx = 2\pi i \sum_{\text{residues}} \text{residues} \text{ of each pole}. \]

Of these, (9.2)

May also derive from trigonometric interpolation, ie Fourier series truncated at least $\frac{1}{2}$. is also exp accurate.
Converse of this holds: Lemma 9.14: If \( x_j \) nodes sat. \( q_j(1) = P_j(1) \), it's a Gauss quad.

\[ \text{pf: recall interpolatory quad. has } \int_{\lambda} f(x) = \int_{\lambda} f(x) \, dx \quad \forall f \in C[\lambda, \lambda] \]

claim each \( p \in P_n \) can be written \( p = L_j p_j + q_j(x) \) for some \( q_j \in P_n \)

\[ \text{why? } p-L_j p_j = 0 \text{ at } x_j, \text{ so } q_j \text{ can have at most } |2n(j) - (n_j)| = n \text{ zeros, \newline} \]

\[ \text{So } \sum_{j=1}^{n} q_j(x_j) = \int_{\lambda} (L_j p_j(x) + q_j(x)) \, dx = \sum_{j=1}^{n} q_j(x_j) p_j(1) \quad \& \text{E.D. since interp.} \]

...elif desired orthog.

So, if we can find \( q_j \), a degree-(n_j) poly, orthog. to \( P_n \), with all roots in \([a,b]\),

we'd like these \( q_j \)s give the nodes!

ORTHOGONAL POLYNOMIALS (useful anyway):

**Lemma 9.15:** If unique seq. \( (q_j)\) w/ \( q_0 = 1 \) & \( q_j(x) = x^k + p(x), p \in P_k \)

which are mutually orthogonal, i.e. \( (q_j, q_m) = 0 \quad \forall m < n \quad \& \quad \text{span}\{q_1, \ldots, q_n\} = P_n \)

\[ \text{Pf: } 1, x, x^2, \ldots \text{ are lin. indy. on } [a,b], \text{ so Gram-Schmidt unique:} \]

\[ q_0 = 1 \]
\[ q_1 = x - \frac{(x, q_0)}{(q_0, q_0)} q_0 \]
\[ q_2 = x^2 - \frac{(x^2, q_1)}{(q_0, q_0)} q_0 - \frac{(x^2, q_2)}{(q_2, q_2)} q_0 \]
\[ \vdots \]
\[ q_n = x^n - \frac{(x^n, q_{n-1})}{(q_{n-1}, q_{n-1})} q_{n-1} - \frac{(x^n, q_n)}{(q_n, q_n)} q_n \]

so there is 1 L.I. element of \( P_n \) (in \( n+1 \)-dim vector space) must span it.

In this you'll need this can be done via 3-term recurrence, i.e. \( q_j \) involves \( q_{j-1} \) & \( q_{j-2} \) only.

*Legendre* poly's (but above not std normalization) is unique seq. of orthog. poly's on \([a,b] \)

w/ unweighted inner product \((f, g)\).

**Lemma 9.16** \( q_n \) has \( n \) simple zeros all in \([a,b]\) (good, so they give a Gauss quad).

\[ \text{Pf: } \forall n \geq 1, \quad q_n + q_0 \quad \text{ie } \int_{\lambda} q_n = 0 \quad \Rightarrow \quad q_n \text{ has } n \text{ zeros } x_1, \ldots, x_n \text{ in } [a,b] \]

\[ \text{supp. } m < n, \text{ then } \int_{\lambda} (x-x_j) q_m = 0 \quad \text{so is } q_m \text{ on } P_{n-1} \quad \text{so is } \quad q_m^2 \text{ contradicts.} \]

but \( \int_{\lambda} q_n = 0 \text{ since } \text{span} \{q_1, \ldots, q_n\} \text{ has fixed sign, not } \equiv 0 \quad \Rightarrow \quad \text{unsolv.} \]

In practice, how compute \( \{x_j\} \)? They are eigenvals of \( B_n \)

& \( x_j \)s come from eigenvals. "Golub-Welsch"

See code gaussin

This is \( O(n^3) \) slow! Chopin-Lin-Reid, has better

\( \text{Golub-Lin-Reid, } O(n) \) idea which is \( O(n) \): find \( x_j \) from \( x_j \) by Taylor expansion, etc.

Given by Legendre poly recurrence.
Periodic numerical quadrature

The simplest rule to approximate \( \int_0^{2\pi} f(t) \, dt \) is sometimes the best: sum \( N \) equally spaced samples of \( f \)!

**Theorem (Davis ’59):** Let \( f \) be \( 2\pi \)-periodic, and *real analytic*, meaning \( f(z) \) is bounded and analytic in some strip \( |\text{Im } z| \leq a \) of half-width \( a > 0 \). Then there is a const \( C > 0 \) (indep. of \( N \)) such that the error is

\[
\left| \frac{2\pi}{N} \sum_{j=1}^{N} f \left( \frac{2\pi}{N} j \right) - \int_0^{2\pi} f(t) \, dt \right| \leq C e^{-aN}
\]

- exponential convergence in \( N \): doubling \( N \) squares your accuracy
  - very desirable: can get accuracies of \( 10^{-14} \) w/ little effort. Carries over to solving the PDE!
Proof

Residue Thm: \[ 2\pi i \sum \text{residues} = \text{closed contour integral in } \mathbb{C} \]

Beautiful cotangent function \( \cot(z) \): poles at \( \pi j, j \in \mathbb{Z} \), residues 1

\( f \) analytic \( \frac{1}{2i} f(z) \cot\left(\frac{N}{2}z\right) \): poles at \( \frac{2\pi}{N} j \), residues \( \frac{1}{iN} f\left(\frac{2\pi}{N} j\right) \)

Res. Thm in strip: \[ \frac{2\pi}{N} \sum_{j=1}^{N} f \left( \frac{2\pi}{N} j \right) = \int_{\Gamma_1 + \Gamma_2} \frac{1}{2i} f(z) \cot \left( \frac{N}{2}z \right) \, dz \]

integrand pure Im on \( \mathbb{R} \), so
Re parts antisymmetric \( \uparrow \) add
Im parts symmetric \( \uparrow \) cancel

\[ = \text{Re} \int_{\Gamma_1} (-i) f(z) \cot \left( \frac{N}{2}z \right) \, dz \]
\[
\frac{2\pi}{N} \sum_{j=1}^{N} f \left( \frac{2\pi}{N} j \right) = \text{Re} \int_{\Gamma_1} (-i) f(z) \cot \left( \frac{N}{2} z \right) \, dz
\]

Cauchy integral formula in \( D_1 \) (since \( f \) analytic):

\[
- \int_{\Gamma} f(z) \, dz = \int_{\Gamma_1} f(z) \, dz
\]

add Re part of this to previous eqn:

\[
\frac{2\pi}{N} \sum_{j=1}^{N} f \left( \frac{2\pi}{N} j \right) - \int_{\Gamma} f(z) \, dz = \text{Re} \int_{\Gamma_1} \left[ 1 - i \cot \left( \frac{N}{2} z \right) \right] f(z) \, dz
\]

error of our quadrature \quad exp. small \leq \frac{2}{(e^{aN} - 1)} \quad \text{bnded in } D_1

QED

- Research: good quadrature schemes for \( f \)'s with singularities?
Integral Equations: given integral \( f_{[a,b]} \), function \( f \) on \( (a,b) \), kernel \( k \) on \( C[a,b]^2 \), solve \( \int_a^b k(t,s) u(s) \, ds = f(t) \quad \forall t \in [a,b] \). **Fredholm 1st kind**.

or 2nd kind \( u(t) + (Ku)(t) = f(t) \quad \forall t \in (a,b) \).

Functional eqns. \( Ku = f \quad \text{involves like} \ A^2 = b \), ie \( \int_a^b f(t) u(t) \, dt = f(\theta) \) \( u \in \text{inner prod} \cdot A \) \( k(t,s) \) & \( u \).

What is \( (K^2u)(t) ? = \int_a^b k(t,s) (Ku)(s) \, ds = \int_a^b k(t,s) \int_a^b k(s,r) u(r) \, dr \, ds \)

when \( K' \) is kernel of \( K^2 \) so \( K'(t, r) = \int_s a(k) \, k(r, s) \, ds \)

\[ \text{[like matrix prod.]} \ A B_{i,j} = \sum_{k} a_{ij} b_{k} \]

- if \( k(t,s) = 0 \) for \( s > t \) **lower-triangular** then \( A \) \( n \times n \) Fredholm \( n \times n \) can be written \( \sum_{i} k(t,s) u(s) \, ds = f(t) \).

\( \text{has unique soln., we want concern} \) \( u \), \( eg \) \( k = 1 \) : \( \int_0^t u(s) \, ds = f(t) \leftrightarrow u(t) = f'(t) \) \( \text{soln.} \).

Fredholm has still on both site of drag:

\( s_0 t^2 s \, u(s) \, ds = \frac{t^3}{3} \quad 0 < t < 1 \).

\( \text{bring out}: \quad t^2 \int_0^t u(s) \, ds = \frac{t^3}{3} \quad \text{so} \quad \int_0^t u(s) \, ds = \frac{t^3}{3} \quad \text{is a soln.} \).

**So highly nonunique, typ. of 2nd kind**.

\( K \) is rank-1 since for any \( u \), \((Ku)(t) = \text{a multiple of} \ t^3 \).

Bounded operator: \( \|K\| = \sup_{Ku \neq 0} \frac{\|Ku\|}{\|u\|} \) for your choice of norm, \( eg \) \( L_2 \), \( L_\infty \), etc.

\[ \|K\| \leq \sup_{t \in [a,b]} |(Ku)(t)| = \sup_{t \in [a,b]} \int_a^b |k(t,s) u(s)| \, ds \leq \int_a^b |k(t,s)| \, ds \quad \text{if} \quad \|u\| = 1 \]

\[ \text{eg.} \quad k \in C \left[ [a,b]^2 \right] \quad \text{has} \quad \|K\| < \infty \quad \text{all} \quad k \in C \left[ [a,b]^2 \right] \]

Can say more: above \( t \) is \( t \) ? Why? (Continuous time \( u \) can approximate \( \text{sign} \ k(t,s) \) arb. well.)

\[ \text{This is NA Thm M.S. (explicitly gives example of this)} \]

\[ \text{pick} \ t_0 \in \text{the t which minimize} \int_a^b |k(t,s)| \, ds \quad \text{exists} \]

\[ \text{eg.} \quad k < t \]
Kernel may blow up on diagonal, e.g. $K(t,s) = \frac{1}{|t-s|^\gamma}$

But if $|K(t,s)| \leq \frac{C}{|t-s|^\gamma} \quad \forall s, \quad 0 < \gamma < 1$ the $L_1$ norm of each row bounded,

$$\Rightarrow \|K\|_1 < \infty \quad \text{norm bounded}$$

called "weakly singular". If $\gamma > 1$ strongly singular, may be unbounded operator.


$$u(t) = \sum_{i=1}^{n} \int_{y_i}^{t} k(t,y) u_i(y) \, dy = \sum_{i=1}^{n} \int_{y_i}^{t} k(t,y) \, dy$$

approx $u$ by $u_n$ which obeys

$$u_n(t) = \sum_{j=1}^{n} w_j \int_{t}^{s_j} k(t,s) u_n(s) \, ds = \sum_{j=1}^{n} w_j k(t,s_j) u_n(s_j)$$

ie $(I - \mathbf{A}) u_n = f$

Thus values at nodes $u_n^{(n)} := u_n(s_i)$ soln. the lin. sys.

$$\sum_{j=1}^{n} w_j k(s_i,s_j) u_j = f(s_i)$$

ie $(I - \mathbf{A}) u_{n}^{(n)} = f$

rhs at nodes vector.

So $u$ solved for at nodes — how get back full func $u_n(s)$? Lagrange interp. pos.; kelley

Thm (12.11) If any vector $\{ u_i^{(n)} \}_{i=1}^{n}$ is soln. to (5), then $u_n(t) = \sum_{i=1}^{n} w_i k(t,s_i) u_i^{(n)}$

solve (6), (exactly), surprisingly that we have exact interpolation.

For: $u_n(s_i) = u_i^{(n)} u_i$ since set $t = s_i$ in (6) $\Rightarrow$ give (5).

Use this to sub for $u_j^{(n)}$ in (n) terms it into (5), ok! Subtle!

(r) express $u$, as $f + \text{span } \{ \text{column slices of kernel at node} \}$

(c) is eqn of Vandermonde sys. requiring interp. agree at nodes; (n) is interp. formula

if drop $f$, can apply to 1st kind, but there is no interp. formula, (n) now, just get $\sum_{i=1}^{n} w_i k(t,s_i)$

Note: $\| K_n - K \| \rightarrow 0 \quad \text{as } n \rightarrow \infty$ not converge in norm topology (is unstable).

But do have ptwise convergence.

$$\| (K_n - K) \phi \| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (\text{can still have stuff})$$
Compact operators — just the essentials: (see [18], [28], [31]) 

1. any non-compact range must behave like finite-dim ops (i.e. square matrices). 

$$X = C([0,1])$$ a topological space, $$f \in X$$ is a point in $$X$$. Choose metric norm $$||f||_2$$, i.e. 

- seq. $$(f_n)_{n \in \mathbb{N}}$$ bounded if $$||f_n||_2 \leq C \ \forall n \in \mathbb{N}$$ — not seq. gone forever a long time? 

- seq. $$(f_n)$$ converges to $$f \in X$$ if $$\forall \epsilon > 0$$ ex. $$N$$ s.t. $$||f_n - f||_2 < \epsilon \ \forall n > N$$ 

Then (Bolzano-Weierstrass) if $$\dim(X) < \infty$$, every bounded seq. contains a convergent subseq. 

$$f_0, f_1, f_2, \ldots$$ converges subseq. also go on forever!

But infinite dim spaces such as $$C([0,1])$$, $$L^2([0,1])$$ all forms $$f$$ s.t. $$\int_0^1 |f(x)|^2 \mathrm{d}x < \infty$$ 

- eg. Fornier seq. $$\left( \sin \frac{n\pi}{2n} \right)_{n \in \mathbb{N}}$$ bounded in $$L^2(0,1)$$ but has no convergent subseq. 

Def.: Linear op. $$K: X \to Y$$ between normed lin. spaces $$X, Y$$ is compact if given any bounded seq. $$(f_n)$$ in $$X$$, the seq. $$(Kf_n)$$ contains a convergent subseq. 

- eg. if $$K$$ has finite-dim range $$\mathbb{R}^N$$: $$BW \equiv (Kx_n)$$ has conv. subseq. $$\Rightarrow K$$ contd. 

But $$K{\text{Id}}$$ in finite-dim space $$Y = X$$: can feed it forward seq. $$\Rightarrow K$$ contd. 

Useful fact: Cont. op. maps unit ball to hyperellipsoid w/ axes $$\lambda_i$$ decreasing to zero: 

1. Cont. ops have non-trivial eigenvalues w/ zero the only limit: $$K\phi = \lambda \phi$$ then $$\lim \lambda = 0$$ 
2. Cont. $$\Rightarrow$$ bounded (easy to prove). 

3. Integral operator w/ weakly singular kernel, $$|k(t,s)| \leq \frac{C}{|t-s|^{N-1}}$$, $$N > 1$$, $$\forall t, s$$, is contd. (in $$C[0,1]$$) 

4. $$K$$ contd. if it is the operator norm limit of seq. $$K_1, K_2, \ldots$$ of contd. ops, i.e. $$\lim ||K-K_n||_2 = 0$$ 

eg. acting on sequence, $$K \{a_n \}_{n \in \mathbb{N}} : \Rightarrow \{0, \frac{1}{2}, \frac{1}{4}, \ldots \}$$ seq. $$\infty$$-0. 

Then, can truncate to finite-dim ops $$K_n$$ 

5. Fredholm Alternative: Let $$K: X \to X$$ be contd. 

Then either: 

i) for each $$f \in X$$, $$(I-K)f = 0$$ has unique soln. iff $$A = 0$$ 

ii) homogeneous (I-K)f = 0 has multiple solns. 

This asserts existence of soln to 2nd-kind IE from uniqueness... amazing! 

Behave like finite linear systems: $$Ax = 0$$ has soln. $$\forall b$$ iff $$A = 0$$ has only the trivial soln. (non-surjective) 

6. $$K$$ contd. $$\Rightarrow$$ convergence rate of Nyström method for 2nd-kind IE is $$\|u_n - u\| \leq C||Ku - Ku_n||_2$$ 

(see [26], [29], [30]).
Lec 9 (M12) part 2: PDEs.

\[ \Delta := \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \text{ in } \mathbb{R}^2 \]

\[ \Delta u = 0 \text{ in } \Omega \Rightarrow u \text{ harmonic} \quad (\Rightarrow u = \text{Re } v \text{ for some } v \text{ analytic in } \mathbb{C}^n) \]

Check \[ \ln \frac{1}{|x|} = -\ln |x| \] obeys \[ \Delta \ln \frac{1}{|x|} = 0 \quad \forall x \neq 0. \]

\[ \frac{\partial}{\partial x_1} \ln |x| = \frac{1}{2} \frac{\partial}{\partial x_1} \ln (x_1^2 + x_2^2) = \frac{1}{2x_1^2} 2x_1 = \frac{x_1}{|x|^2} \]

\[ = \text{Fundamental Soln. } \Phi(x, y) := \frac{1}{2\pi} \ln \frac{1}{|x-y|} \text{ obeys } \Delta \Phi(x, y) = 0 \quad \forall x \neq y \]

(Note: already seen in quantum stuff in \( \mathbb{C} = \mathbb{R}^2 \).)

Consider shifting the spike to sit at loc. \( y \).

Divergence Thm: \( \vec{a} = (a_i(x)) \) vector field (eg \( a_1, a_2 \in C^1(\Omega) \)) \( \Omega \) may have corners.

Thus \[ \int_{\Omega} \vec{a} \cdot d\mathbf{x} = \int_{\partial\Omega} \vec{n} \cdot d\mathbf{s} \quad \text{ where } d\mathbf{s} \text{ is arc length measure on } \partial\Omega \]

\[ \text{Flux} = \int_{\partial\Omega} \vec{n} \cdot d\mathbf{s} \]

(Choose \( \vec{a} = \vec{U} \vec{V} \) where \( u, v \) scalar func. & prod rule: \( \nabla \cdot (\vec{U} \vec{V}) = u \Delta v + \vec{v} \cdot \nabla v \) check.

Observe, \( \text{curl free} \) \( \int_{\partial\Omega} \vec{n} \cdot d\mathbf{s} = 0 \quad \text{note } \vec{n} \cdot \vec{V} \mid_{\partial\Omega} := \text{Un normal deriv.} \)

Directional deriv of Found Soln: say \( \vec{n} \) is a unit vector in direction.

Deriv of \( \Phi(x, y) \) with moving source pt. \( y \) in \( \vec{n} \) direction:

\[ \frac{\partial \Phi}{\partial y} = \frac{1}{2\pi} \vec{n} \cdot \nabla \ln \frac{1}{|x-y|} \]

\[ \frac{\partial}{\partial y} \ln \frac{1}{|x-y|} = -\frac{1}{2} \frac{\partial}{\partial y} \ln |x-y|^2 = \frac{1}{2} |x-y|^2 \frac{\partial}{\partial y} [-(x-y)^2 + (x-y)^2] = -\frac{x-y}{|x-y|^2} - 2(x-y)|x-y|^2 \]

So \( \frac{\partial \Phi}{\partial y} = \frac{1}{2\pi} \vec{n} \cdot \frac{(x-y)}{|x-y|^2} \)

is harmonic for \( x \neq y \).

Last time:

Let \( u = \frac{1}{r} \) in \( \mathbb{R}^3 \), \( r = \sqrt{x^2 + y^2 + z^2} \), \( \Delta u = 0 \), \( \nabla u = -\frac{1}{r^2} \hat{r} \).

This gives the potential of a point source.

Green's Representation Formula: Let \( u \in C^2(S) \) be harmonic in \( S \), then

\[
\int_S \Delta u \, dS = \int_{\partial S} u \nu \, ds.
\]

Since \( u = 0 \) on \( \partial S \), we have

\[
\int_S \Delta u \, dS = 0.
\]

So, \( u = 0 \) is the harmonic function that satisfies the boundary conditions.

Now, let's consider the case where \( S \) is a sphere.

For \( S \) a sphere of radius \( r \) centered at \( x \),

\[
\Omega = S \setminus B(x; r) = \{ y \in S : |y - x| < r \}
\]

where \( B(x; r) \) is the ball of radius \( r \) centered at \( x \).

So, \( u \) is harmonic outside the sphere.

Next, we consider the case where \( S \) is a sphere of radius \( R \) centered at \( x \),

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\Omega = S \setminus B(x; R) = \{ y \in S : |y - x| > R \}
\]

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Finally, we consider the case where \( S \) is a sphere of radius \( r \) centered at \( x \),

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\Omega = S \setminus B(x; r) = \{ y \in S : |y - x| < r \}
\]

where \( B(x; r) \) is the ball of radius \( r \) centered at \( x \).

So, \( u \) is harmonic outside the sphere.

In conclusion, \( u \) is harmonic in \( S \) and satisfies the boundary conditions.

\[ \square \]
Case \( x \in \Omega \)
(sketched)

\[
\nabla' \quad \text{new } \Omega' = \Omega' + \epsilon \Omega' \quad \text{where } \epsilon \Omega' = \epsilon \Omega(x; \epsilon) \cap \Omega
\]

so \( \lim_{\epsilon \to 0} \epsilon \Omega' \) \( \Omega' \) locally flat so \( \epsilon \Omega' \to \) half-circle

\[
\frac{1}{2\pi} \ln \left( \int_{\partial \Omega} u(y) \, dy \right) \to 0 \quad \text{since } u \text{ bounded, } \epsilon \ln \epsilon \to 0.
\]

\( \Omega' \to \Omega \).

Case \( x \in \mathbb{R}^n \setminus \bar{\Omega} \).

There is no ball, \( \Omega = \Omega \).

Useful corollaries

1) Since \( \partial \Omega \) is analytic, \( u \) is analytic in \( \Omega \) w/ each \( \epsilon \Omega \), regardless how nonsmooth \( \partial \Omega \) is. (The 6-8)

2) Mean val. thm. for harm funs. (UE Thm 5.7) if \( u \) harm, \( u(x) = \frac{1}{2\pi} \int_{\partial \Omega} u(y) \, dy \quad \text{for } \partial \Omega, \) then

\[ \nabla \Delta u = 0 \quad \text{in } \Omega \]

has at most one soln. Suppose \( u_1, u_2 \) solns, then \( w = u_1 - u_2 \) harm.

\[
\text{Max. Principle: harm funs attain max. min. on } \partial \Omega.
\]

ii) \( u = f \) on \( \partial \Omega \)

iii) \( \int_{\partial \Omega} u(y) \, dy = \int_{\partial \Omega} -1 \quad \text{other cases by Max. Princ.} \)

\[ \text{Gauss' Law} \] (CL) can also prove via ZF & small balls directly (fig 11).

Layer potentials (new notation)

\[ \Phi \in C(2\Omega) \]

\( (S\Phi)(x) = \int_{\partial \Omega} \Phi(x, y) \, dy \quad \text{for single-layer potential} \)

\( (D\Phi)(x) = \int_{\partial \Omega} \frac{\Phi(x, y)}{\|x - y\|^n} \, dy \quad \text{for double-layer potential} \)

Often use \( S\Phi \) for SLP density

\[ \epsilon \text{ GRK says, in } \Omega, \quad u = S\Phi + D\Phi \quad \text{where } \Phi = u_n \text{ on } \partial \Omega \]

eg. \( \text{CL says, } D\Phi \text{ generates potential } -1 \text{ in } \Omega, \ 0 \text{ outside.} \) - test in HW5.
How well such integrals in practice? Change variable:

\[
\begin{align*}
\frac{d}{ds} \int_{2\pi}^s \frac{dz(s)}{z(s)} \, ds &= \int_{2\pi}^s \frac{dz(s)}{z(s)} \, ds \\
\text{say } z(s) \text{ parameterize } \gamma, \quad z(2\pi) = z(0) \quad \text{i.e. } z: [0, 2\pi] \to \mathbb{R}^2 \\
\text{since } (z_1(s), z_2(s))
\end{align*}
\]

\[g(2\pi) \text{ given by } f(s) \text{ in polar: } z_1(s) = f(s) \cos s, \quad z_2(s) = f(s) \sin s
\]

Then \[
\int_{2\pi}^s g(s) \, ds = \int_0^{2\pi} g(z(s)) |z'(s)| \, ds
\]

quadrature via periodic trap rule

\[
\frac{2\pi}{N} \sum_{j=1}^N g(z(s_j)) |z'(s_j)|
\]

"speed function"

At each surface node \(z(s_j)\) also need normal \(n_j = \text{"z' rotated CW 90 & normalized"}\)

\[
n_j = (n_1, n_2) = \frac{1}{|z'(s_j)|} (z_2(s_j), -z_1(s_j))
\]

Coding: recommend you set up \(f()\) and \(g()\) then pass the function to routine that

\[
\text{H} \text{Havik solver - see for Nyström}
\]

\(1)\) plots potential due to given density \(f()\) and

\(2)\) uses DCP

Jump relations: note DCP: DCP has jump in value as \(x\) crosses \(\gamma\) from inside to outside.

Define \(\text{let } x \in \mathbb{R}^n\)

\[
U^+(x) := \lim_{h \to 0^+} U(x \pm h\hat{n}_x)
\]

\[
U^-(x) := \lim_{h \to 0^-} U(x \pm h\hat{n}_x)
\]

\(\text{jump}\) at \(x\) \(\text{normal at } x:\)

\[
\begin{align*}
U^+(x) &= U^-(x) \quad \text{and } U = S 0, V = D 1
\end{align*}
\]

The expected DCP has \(U^+(x) - U^-(x) = \tau(x)\), true.

Thin (TR's) Let \(\gamma\) be \(C^2\) \(\text{ie } z_1, z_2 \in C^2\), \(\sigma, \tau \in C(\gamma)\), and \(U = S\sigma, V = D\tau\)

Then for \(x \in \gamma\)

\[
U^+(x) = \int_{\gamma} \Phi(x, y) \sigma(y) \, dy
\]

\[
U^-(x) = \int_{\gamma} \frac{\partial \Phi(x, y)}{\partial n_x} \sigma(y) \, dy + \frac{\sigma(x)}{2}
\]

\[
V^+(x) = \int_{\gamma} \frac{\partial \Phi(x, y)}{\partial y} \tau(y) \, dy + \frac{\tau(x)}{2}
\]

\[
V^-(x) = \int_{\gamma} \frac{\partial \Phi(x, y)}{\partial y} \tau(y) \, dy
\]
Batch integral ops: 

- \( S: C(\mathbb{R}) \to C(\mathbb{R}) \) has kernel \( \Omega(x,y) \) - note: weakly singular (\( \text{div} y = y \to \infty \)).

- \( D: C(\mathbb{R}) \to C(\mathbb{R}) \) kernel \( \frac{\delta^2}{\delta y^2} \)

So \( JK \)'s says, \( v^\pm = D\chi \pm \frac{1}{2} \chi \)

Say want \( v \) to solve BVP: \( \begin{cases} Dv = 0 & \text{in } \lambda, \\ v = f & \text{on } 2\lambda \end{cases} \) already set: by reg. \( v = D\chi \) !

\( \Rightarrow (I - \frac{1}{2}) \chi = f \) Fred.IE, (which kind? 2nd due to \( \frac{1}{2} \)), a batch IE (BIE)

or \( (I - 2D) \chi = -2f \) BIE in std. 2nd kind form.

Recipe to solve BVP: i) solve BIE for \( \chi \), ii) reconstruct \( v = D\chi \) in interior of \( \lambda \).

Then: let \( \lambda \) be \( C^2 \) smooth, \( \Gamma \)-periodic. Then, kernel of \( D \) is continuous.

\( C^2 \) means \( \Omega(x,y) \) are circles, local curvature of \( \lambda \)-dictates which contour you're on. \( \Rightarrow \) curvature needs to be cont. \( \Rightarrow C^2 \).

Proof:

1. parametrize by \( z: \mathbb{R} \to \mathbb{R}^2 \), \( 2\pi \)-periodic. \( f \in C^2 \) means \( \frac{\partial^2 z(t)}{\partial t^2} = \frac{\partial z(t)}{\partial t} \) both cont.

2. Since \( \Gamma \) is the period of \( z \), \( \Gamma \) is the period of \( f \).

3. For \( t, s \in [0,2\pi] \), kernel \( k(t,s) = \frac{1}{2\pi} \frac{n(s) \cdot (z(t) - z(s))}{|z(t) - z(s)|^2} \)

4. \( \frac{\partial}{\partial t} \) top & bot. cont. wrt \( s \), \( t \partial s \) cont. for \( s \neq t \).

5. \( \lim_{t \to s} k(t,s) \) top & bot. vanish \( \Rightarrow \) l'Hôpital: \( \frac{\partial}{\partial t} \) top = \( n(s) \cdot \frac{\partial z(t)}{\partial t} \) \( \to 0 \) also! Why?

\( \frac{\partial}{\partial t} \) btm = \( 2 \frac{\partial z(t)}{\partial t} \cdot \frac{\partial z(t)}{\partial t} \)

\( \frac{\partial}{\partial t} \) btm = \( 2 |\frac{\partial z(t)}{\partial t}|^2 \Rightarrow 2 \frac{\partial z(t)}{\partial t} \)

Combine: \( \lim_{t \to s} k(t,s) = \frac{1}{2\pi} \frac{n(s) \cdot \frac{\partial z(t)}{\partial t}}{|z(s)|^2} = -\frac{K(s)}{2\pi} \)

\( K = \text{local curvature} = \frac{1}{|\text{curv. radius}|} \)

In practice BIE all done wrt \( s, t \in [0,2\pi] \), by changing one length \( ds, dy \) to \( |\frac{\partial z(t)}{\partial t}| \)

\( f, \chi \) are funs on \( [0,2\pi] \), and \( D \) has kernel \( k(t,s) = \frac{1}{2\pi} \frac{n(s) \cdot (z(t) - z(s))}{|z(t) - z(s)|^2} \cdot z(s)^t \).

\( \text{Note: } \)
or on the diagonal, \( k(s, s) = \frac{1}{4\pi} K(s) \cdot |z(s)| \rightarrow \) in code as \( \frac{v(s) \cdot z''(s)}{|z'(s)|^2} \)

Solving via Nyström \((I - A) \bar{c} = -2f\text{ lin. sys.}\),

where \( \text{INM} \), \( A \) has entries \( a_{ij} = s_i + 2K(s_i, s_j) w_j \quad s_j = \frac{2\pi j}{N} \)

col. vec. \( f \) has \( f_j = f(z(s_j)) \), the body data at the nodes.

Solution vector \( \bar{c} = \{c_j\}_{j=1}^N \) is density at nodes.

Commonly, \( v = Dc \) is then approximated using these same quadrature nodes.

Interior soln in \( \Omega \),

... this isn't always accurate!

(Active research by us!)

Then: above BVP has a soln. \( Pf: D \text{ kernel cont.} \Rightarrow D \text{ cont.} \)

Can show \( D \) doesn't have \( i \) as an eigenvalue.

\( \Rightarrow \) by Fredholm Alternative, soln \( \exists \Rightarrow \) soln \( v \) exists

Proof of JK3 (back): need to show \( v = Dc \) can be continuously extended from \( \frac{\sqrt{v^+}}{\sqrt{v^+} - 1} \) to \( \Omega \) w/ lim. value \( -v_+ = (D+\frac{1}{2})c_+ \)

\( \Rightarrow v = (D+\frac{1}{2})c \)

1. Split into GL correction: let \( v = z + h v_{\text{cor}} \)

\[ v(x) = \int \left( \sum_{j=0}^{\infty} \frac{\partial^0 y(x, y)}{\partial x^0} \right) d\gamma_j + \int \left( \sum_{j=0}^{\infty} \frac{\partial^0 y(x, y)}{\partial x^0} \right) (\tau(y) - \tau(z)) d\gamma_j \]

by GL \( \Rightarrow \lim_{y \to z} \int_{\Omega} v(x, y) dx = \lim_{y \to z} \int_{\Omega} \tau(y) d\gamma = \tau(z) \)

\[ \Rightarrow v = \tau(z) \]

If \( \lim_{y \to z} v(z, y) = v(z, 0) \), uniformly in \( z \in \Omega \), we are done, since GL accounts for the jump.

2. Pick radius \( r > 0 \) & split \( y \)-integral into far \& local parts:

\[ v(z) - v(z, 0) = \int_{|y-z| \geq r} \left( \sum_{j=0}^{\infty} \frac{\partial^0 y(x, y)}{\partial x^0} \right) (\tau(y) - \tau(z)) d\gamma_j + \int_{|y-z| \geq r} \left( \sum_{j=0}^{\infty} \frac{\partial^0 y(x, y)}{\partial x^0} \right) (\tau(y) - \tau(z)) d\gamma_j \]

Assume \( |\tau| \leq c \), then \( |\tau| \leq c \|	au\|_2 \int_{|y-z| \geq r} \left| \sum_{j=0}^{\infty} \frac{\partial^0 y(x, y)}{\partial x^0} \right| d\gamma_j \), so \( |\tau| \leq \|	au\|_2 \)

Lemma (local part): \( \exists h > 0 \text{ & } C \text{ ind. of } h \) s.t. \( \int_{|y-z| \leq r} \left| \sum_{j=0}^{\infty} \frac{\partial^0 y(x, y)}{\partial x^0} \right| d\gamma_j \leq C \quad \forall r \leq \frac{h}{2} \)
Then \[|v(x)+v(x, h)| \leq C \frac{h^2}{2} + 2C \max_{z \in \mathbb{R}} |\tau(z)-\tau(x)| \]

Given \( \varepsilon > 0 \) can choose \( r > 0 \) so \( \frac{\varepsilon^2}{4C} \)

since \( \tau \) cont., \( \Rightarrow \) uniform cont.

Then choose \( h_0 = \frac{\varepsilon r^2}{2C} \)

\[ |v(x)+v(x, h)| \leq \varepsilon \quad \forall x \in \mathbb{R} \land \text{all } h < h_0, \text{ such that } \exists h_0 > 0 \text{ exists for each } \varepsilon > 0 \]

\[ v(x, h) \to v(x) \text{ uniformly} \]

Finally, pf of LPL:

1. Normal lemma (NL): \( \exists L \text{ st. } |v_{xy}(z, y)| \leq L |z-y|^2 \quad \forall z, y \in \mathbb{R} \)

2. Taylor thin w/ \( s \), param.

Then \[ |v(x, h)| \leq \frac{|v_{xy}(z, y)|}{2\pi |x-y|^2} + \frac{|v_{z}(y, x, z)|}{2\pi |x-y|^2} \]

\[ \leq C + C \frac{b}{|x-y|^2 + h^2} \]

where \( b < \frac{1}{2} \), \( \forall y \in \mathbb{R} \).

Thus patch of around \( z \) where \( v_{xy}(z, y) \geq \frac{1}{2} \), \( \forall y \in \mathbb{R} \).

Patch means \( \Gamma = \{ z, y : |z-y| < r, y \in \mathbb{R} \} \).

Can project onto line \( y = x \), tough factor 2 in variable change \( ds \to ds \)
Prove JR3 from loc 11.

Other BVPs.

Ward's interior Dirichlet: "what temperature does a uniformly conducting body settle to when it starts from temperature $f$?"

Interior Neumann: \(\Delta u = 0\) on $\partial \Omega$ \(\text{equilibrium temp. dist.} f\) \(\text{specified heat input}\) \(\text{flux at each pt. on } \partial \Omega\).

What if pumping in more heat than extracting? Blow up!

Already know a 'harm. EQ': $\int_{\Omega} \nabla \cdot \nabla u = 0$ for $\exists$ $f$ for existence.

Then if $u_1, u_2$ are solns, $w = u_1 - u_2$ sat $\Delta w = 0$ in $\Omega$ \(\nabla \cdot \nabla w = 0\) on $\partial \Omega$ \(\text{we const. a soln.}\)

The only soln: $\Delta u = \lambda u$ \(\text{not triv.}\) $u = \lambda u$ $\lambda$ Neumann eigenvalue.

The solution: if we choose $u = DT$ as before, $BC_1$ is $\int_{\Omega} \Delta u \cdot \nabla \phi d\Omega = 0$ (mixed DLP: hyperbolic)

instead try: $u = \int_{\partial \Omega} \frac{\partial u}{\partial n} \phi dl$, so $\int_{\Omega} \Delta u \cdot \nabla \phi d\Omega = 0$

IE is $\text{IE is } (I + 2D^2) u = 2f$. 2nd kind again. $\specialcell{\text{Det } \Rightarrow \text{Det opt}}$

But we know nonunique since BVP has $\lambda$ but in practice, backward stable (im. solver should give $u$ here). Can prove we get $C^{1,1}$ + cond of $\Delta u$. $\Delta u$ which satisfies $\Delta u = f$ which satisfies $\Delta u = f$

Afternoon book: solve $\text{IE is } (I + 2D^2) u + \sigma(x) = 2f$ solves $\Delta u = f$.

Prove uniquely solvable $\forall f \in C(\partial \Omega)$.

Extend BVPs: eg Dirichlet. $\Delta u = 0$ in $\Omega \setminus \partial \Omega$.

has unique soln $\forall f \in C(\partial \Omega)$.

Follows from $\hat{u}(x) = \tilde{u}\left(\frac{x}{|x|}\right)$ "Kelvin transform of $u"$ maps $\mathbb{R}^2 \setminus \Omega$ to $\mathbb{R}^2 \setminus \Omega$.

It is harmonic in $\mathbb{R}^2 \setminus \Omega$. $\tilde{u}$ is harmonic in $\mathbb{R}^2 \setminus \Omega$.

Condition at $\partial \Omega$ is imposed to "match data at $\partial \Omega$."

Harmonic at $\partial \Omega$.

Physically this indeed since potential at a node needs to be specified.
Close this today.

Let's look at the Dirichlet BVP
\[ u(x) = f(x) \]
(proven unique)

Let's look at the Neumann BVP
\[ \int_{\partial \Omega} \nabla u \cdot n = 0 \]
(proven unique up to a constant, needs \( \int_{\Omega} f = 0 \))

These are both "indirect" BIEs: pick a representative for soln. \( u \) as LP, so that BIE for unknown density comes out 2nd kind.

Why not 1st kind? try \( u = \delta \) for int. Dir., want \( BC \)
\[ \delta = 0 \text{ for int. Dir.} \]

but \( \delta \) equals accumulating at zeros, is ill-conditioned in a bad way (for \( N \) large, use iteration rather than \( O(N) \) direct (in solvers, they hate such a matrix).

"Direct" BIE also possible: GRF in interior, \( x \in \Omega \) then
\[ (S u - D u(x)) = u(x) \]  \( \text{for} \ x \in \Omega \)

Take \( x \to \partial \Omega \) use JR2 & 3, get
\[ (I + 2D) u = u \]
\[ \text{say you want to solve int. Neumann BVP} \]
\[ \text{then } u - f = 0 \text{ so } RICS SF known} \]

As here, direct gives adjoint of indirect:
the unknown isn't a density, rather the value.

When BIE solved, use (x) to reconstruct \( u \) on \( \partial \Omega \).

Note: since we know homog. int. Neumann BVP has only const solns, i.e. \( u = \text{const} \) then \( \text{Null}(I + 2D) = \{ \text{const functions} \}

Exterior prob: e.g. Dirichlet BVP: \( \Delta u = 0 \) in \( \mathbb{R}^3 \)
\[ u = f \text{ on } \partial \Omega \]
\[ u \text{ bounded as } |x| \to \infty \]
\[ \text{extra condition needed for uniqueness}. \]
(physically: your total charge on body)
\[ \text{let } \phi(z) = u(z) \]
"Kelvin form of \( u \) turns outside in, yet is harmonic too: now on unbounded domain \( \rightarrow \) existence unique.

Indirect BIE:
\[ u = D x ; \]
JR2 gives
\[ (I + 2D) u = u \text{ is } \delta \]
\[ \text{ie } (I + 2D) = f \]
\[ \text{signs differ from int. Dir. BVP, that's all...} \]
Worse, BIE has no soln. for certain \( f \), even though BVP does have (unique) soln. (Helmholtz)

For suppose \((I + 2D)T = 2f\)

then inner prod. \((\phi, (I + 2D)\phi) = 2(\phi, f)\)

more over

\[(I + 2D)\phi = 0\]

\[\phi \in \text{Null}(I + 2D)\]

Helmholtz eqn.

\[(k^2 + \Omega^2)u = 0\]

plays role of Laplace op

homog. int. Dir BVP \( (\Delta + k^2)u = 0 \) in \( \Omega \)

\( u = 0 \) on \( \partial \Omega \)

there exist discrete \( k_1 < k_2 < k_3 < \ldots \) st. has nontriv. soln.

pf:

\( \Delta u = k^2 u \)

has set discrete Dirichlet eigenvalues \( k_j^2 \), accum only at \( \infty \).

To solve int. Dir BVP \( (\Delta + k^2)u = 0 \) in \( \Omega \)

set \( u = f \) on \( \partial \Omega \)

proceed as Laplace, but need kernel; that's it!

kernel:

\[ \mathcal{D}(x,y) = \pi \frac{H_0^{(1)}(k|x-y|)}{1-x-y} \]

only sing. Hankel function of \( k \), special case, see DLNF.

Asymptotics:

\[ \mathcal{D}(x,y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} + o(1) \]

in same singularity as Laplace \( \Rightarrow \) same JKs!

\[ H_\nu^{(1)}(z) = \frac{1}{2\pi i} e^{iz} \left[ \frac{z^\nu}{\sqrt{z}} - \frac{\nu}{z} \right] + o(z) \]

as \( z \to \pm \infty \)

where from? say \( u(r, \theta) = f(kr) e^{i\theta} \) polar of var. fix \( r \in \mathbb{Z} \) \& \( f(2) \) st. \( u \) sat. Helm. Eqn.

\( 0 = (\Delta + k^2)u = \Delta [f'' + f'] + \frac{1}{r^2} \partial_r [r^2 (\partial_r u - k u) = \frac{k^2}{r^2} f'' + k^2 f'] + (i\nu + k) f' \]

get to eqn.

\[ z^2 f'' + zf' + (z^2 - \nu^2) f = 0 \]

Bessel eqn. (higher order), \( H_\nu^{(1)}(z) \) is soln. to ODE

\\( \nu \) certain asymptotic
Ext Dir BVP:

for $u^s$,

$$0 + \nabla^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \Omega$$

$$u^s = f \quad \text{on} \quad \partial \Omega$$

$$\lim_{r \to \infty} r^{-\frac{d-1}{2}} \left( \frac{\partial u^s}{\partial r} - i ku^s \right) = 0$$

(ie in $d=2$, $u^s \sim o(1/r)$)

No unique soln. $\forall f \in C(\partial \Omega)$, Cauchy-Kovalevskaya Thm 3.7.

Scattering:

say "incident-wave" $u^i : \mathbb{R}^d \to \mathbb{C}$, e.g. $u^i(x) = e^{ik|x|} e^{-i\omega t}$

then if $u^i$ solves $\Delta u^i = 0$ in $\mathbb{R}^d$,

$$-\nabla^2 u^i = \frac{f}{2\pi} \quad \text{in} \quad \mathbb{R}^d$$

why? $u^s = \frac{f}{2\pi}$, cancelling the inc. wave.

Note: $u^i$ doesn't sat. radiation cond, but new waves due to obstacle ($u^s$) do.
Where do Hankel primes come from? \( u(x, t) \theta(y) = 0 \) for \( \forall x, y \).

where \( y > 0 \), call \( \theta = \Theta(y) \), want sat. helm eqn.

\[
\Theta(r, \Theta) = f(kr) e^{i\Theta} \quad \text{polar sep. of var., } r \in \mathbb{Z} \text{ so single-valued, solve for } f:
\]

\[
0 = (\Delta + k^2) f = \frac{1}{4\pi} \int_{\partial D} h_{BR} \partial f + k^2 f = \left( k^2 f'' + \frac{1}{r} f' \right) e^{i\Theta} + \left( \frac{\partial}{\partial \theta} f' + \frac{\partial}{\partial \phi} \right) e^{i\Theta} \quad \text{cancel } e^{i\Theta}
\]

cancel \( e^{i\Theta} \)

\[
\begin{align*}
\Delta f - k^2 f &= 0 \\
\text{Bessel eqn., order } \nu (ODE) \\
H_{\nu}^{(1)} \text{ is soln. if } \nu \in \mathbb{Z} \text{ so } \nu = 0 \text{ solves } \text{order } \nu
\end{align*}
\]

large argument: \( H_{\nu}^{(1)}(z) = \frac{1}{2\pi i} e^{z/2} \left( z - \frac{\nu^2}{4} \right) e^{-2iz/2} = 0 \left( z \to \infty \right) \quad \text{paved in Colloq. 862.}
\]

Also, solutions regular at \( \nu = 0 \) \( J_0 \) Bessel function.

Having non-unique solutions BUE for small \( \nu \).

In 1946 year saw eqn bue bue hampered by ghost of complementary bvp: let \( u = \Delta u \), sat. helm in \( \mathbb{R}^3 \).

solves eqn bvp if \( (1 + i \Delta )u = 2f = -2u \text{, loss from src. field.} \)

will need \( \phi \phi = 1 \) (obtain), same as \( \phi \) plane.

\[
\begin{align*}
\text{suppose } \phi \text{ sat. } (1 + i \Delta )\phi = 0 \text{ in } \Omega \\
\phi = 0 \text{ on } \partial \Omega
\end{align*}
\]

then \( \phi \) is harmonic neumann eigenvalue. ("acoustic resonance of cavity" or \( k^2 \) its eigenvalue.

then by \( \phi \phi = 1 \) (obtain), same as \( \phi \) plane.

\[
\begin{align*}
\text{then } \phi \phi - \Delta \phi &= 1 \text{ in } \Omega \\
\text{take } r \to \infty \text{ & use } J_0 \text{ \( J_0 \) boundary, } \phi = 0 \text{ \( J_0 \) boundary.}
\end{align*}
\]

since \( \phi \) harmonic. (otherwise \( \phi \in \mathbb{C} \) by \( \phi \phi = 1 \))

\[
\begin{align*}
\text{false eqn } \text{null of } (1 + i \Delta )\phi = 0 \text{, singular, not solvable}.
\end{align*}
\]

Show evol recall eqn of \( \nabla \) vs. \( \kappa \): when first \( \kappa = 0 \), then \( \nabla \) (Nec. eqn of \( \nabla \) vs. \( \kappa \))

\[
\begin{align*}
\phi \phi = (\nabla - i \kappa S)\phi = 0 \quad \text{wts. to show } \kappa = 0.
\end{align*}
\]

\[
\begin{align*}
\text{External work, Levi-Civita, 60s.} \quad \text{solve eqn bvp if } (I + 2D - 2iyS)\phi = 2f
\end{align*}
\]

\[
\begin{align*}
\text{and } \partial \Omega = \text{SR3 + SR4 (no jump for } S \text{, unlimited.)}
\end{align*}
\]

\[
\begin{align*}
\text{then } I + 2D - 2iyS \text{ injective } \forall \kappa > 0
\end{align*}
\]

\[
\begin{align*}
\text{pf: let } T \text{ solve } (I + D - iS)\phi = 0 \quad \text{wts. to show } \kappa = 0.
\end{align*}
\]

\[
\begin{align*}
\text{assume evol potential } \nu := (\nabla - iS)\phi \quad \text{then } \nu \phi = 0 \text{ by construction of } BUE \quad (2f \equiv 0)\n\end{align*}
\]
\[ \Rightarrow V = 0 \text{ in } \Omega \setminus \Gamma \text{ by uniqueness of } \text{ext. Dir. BVP for quasiparabolic solv.} \]
\[ \Rightarrow \nabla V = 0 \text{ on } \partial \Omega. \]

\[ \Rightarrow \begin{cases} V_1, V_2 \Rightarrow V^- = -C \quad \text{on } \Gamma_{2,4} \\ V_3, V_4 \Rightarrow V^+ = -i\gamma C \end{cases} \]

\[ \text{Simplify } \int_S : \quad \int_S \nabla V \cdot \nabla V \, ds = \int_{\partial \Omega} \nabla V \cdot \nabla V \, ds \]
\[ \quad + i\nu \int_{\partial \Omega} \nabla V \cdot \nabla V \, ds. \]

\[ \text{Take Im part: } \gamma = 0. \quad \text{QED.} \]

But complex \( k \) mutes this up.

**Notes:**

i) Call such a scheme robust since provably never fails; similar exist for Neumann BVP, transmission, etc.

ii) Quadrature of BIEs now harder: \( S \) has log singularity near diagonal.

Approaches:

a) use correction of periodic trig. rule weights by quasi diagonal, \( \text{ie } j \& k \) integrate small + log \( |s-h| \) smooth to high order.

Kapur-Reddy, 91.

b) find exact weights to integrate \( \log \) smooth globally, product quadrature. Kress, 91.

Better but more analytic work. E.g. Kress, 91.

Projects:

c) other ways to avoid near singularity using new set of nodes. Johnson 99.

These also make \( P \) quadratic high order (\( \text{Helm.:} \) retained only 9th order, unlike Laplace \( k=0 \) circular expansion).

**Fast algorithms:**

- \( \text{eg } N \approx 10^6 : \) can't even fill Nyström matrix \( A(10^6 \times 10^6 \text{bytes } = 10^6 \text{GB}) \)

  - 
    - let alone do dense linear solve (\( N^3 \approx 10^{18} \) steps) \( Ax = b \)

  - Instead: iterative methods, e.g. GMRES (NLA Ch. 33), each step involves \( \text{Ax} \)

  - Converges, stop when residual order \( \| Ax - b \| \) small enough for you.

  - For well-conditioned 2nd kind \( \mathcal{L} \), take only 10-20 steps to get 10 or 11 digits (10-14 accuracy).

  - But 1st kind, terrible convergence rate, worse.

  - So now, whole scheme to solve for \( \mathcal{L} x \) is \( O(N^2) \) since \( x \rightarrow Ax \) is.

  - Can we apply the Nyström method to a vector \( \mathcal{L} x \) faster than \( O(N^3) \)? Yes!
Let \( y_i \in \mathbb{R}^n \) be set of nodes.

The problem: \( A \) has elements \( a_{ij} = \sum \frac{1}{|y_i - y_j|} \).

This is off-diag part of Nyström matrix for \( S \), spectral (eigenvalues), without weights \( w_j \).

run lowrank curve \( m \). \( w_i \seteq N = 1.2 \), \( \nu \seteq 2.3 \).

Low rank requires source - target separation.

A low rank rank means \( \tilde{A} = P Q = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \tilde{A}_{ij} P_{ii} Q_{jj} \).

Eq. via SVD (but that's too slow in practice).

Fix a block, call it size \( N \times N \): source \( y_j \), \( j = 1 \ldots N \), target \( z_i \), \( i = 1 \ldots N \).

Wish to compute \( u_i = \sum_{j=1}^{N} \frac{1}{|y_i - y_j|} = (A^2)^{ij} \).

Charge strength at each node.

Potential due to sources \( U(z) = \sum_{j=1}^{N} \frac{x_j}{|y_j - y|} \) harmonic for \( z \neq y_j \), \( j = 1 \ldots N \).

Goal is equal \( U \) & target \( z_i \), \( i = 1 \ldots N \).

Then (multiple expansion)

outside a disc \( B \) centered at \( O \), containing all \( y_j \) 's,

we can write

\[
W(r, \theta) = C \ln \frac{1}{r} + \sum_{n=1}^{\infty} \left( \cos n \theta - b_n \sin n \theta \right) r^{-n}
\]

or \( u(z) = \ln |z| \), \( u(z) = \sum_{n=1}^{\infty} c_n z^{-n} \).

Thus sums abs. converge in \( \mathbb{R}^n \).
last time empirically observed low-rank property of off-diagonal blocks, but how exploit this (without decaying SVD which is $O(1)$) 

last time: want real potential $u(z) = \sum_{j=1}^{n} y_j \ln \frac{1}{r^2 - y_j}$ at $z = z_i$, $i = 1 \ldots N$. 

$\mathbb{R}^2$: $y_j \rightarrow \mathbb{C}$, $z_j \rightarrow \mathbb{C}$.

source

targets
w/ strengths $y_1 \ldots y_N$. 

since $N$ sources, $N$ targets, $N^2$ interactions (more to recall ln dist $N^2$ times, namely)

- if $u_i := u(z_i)$, $t_i := A_i x_i$, where $A_i$ is some $N \times N$ off-diagonal block of $\Sigma N \times N$ jordan matrix.

Note: $u(z)$ harmonic for $z \neq y_j$, $j = 1 \ldots N$.

The (multipoles expansion): let $B$ be a disc containing all $y_j$'s, centered at $0$, radius $R$, 

\[ u(r, \theta) = C_0 \ln \frac{1}{r} + \sum_{n=1}^{\infty} \left( a_n \cos n\theta + b_n \sin n\theta \right) r^{-n} \]

\[ \text{sums abs. convergent in } r > R. \]

Note: solving $C_0 = 0$, true for any, func $u$ harmonic in $r > R$ iff $u(0) \equiv o(1)$ as $r \to 0$.

Or considering $\mathbb{R}^n = \mathbb{C}$, $u(z) = \Re \left( C_0 \ln \frac{1}{z} + \sum_{n=1}^{\infty} C_n z^{-n} \right)$

Lowent expansion (Taylor expnd.)

Say truncate to $p$ terms, how bad is error?

Consider single unit charge @ $y$: $u(z) = \ln \frac{1}{z-y}$, let's work w/ complex-valued potential, take $\Re$ at end.

$= \ln \frac{1}{z} = \ln (1 - y) \quad \text{Taylor} \quad \ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \quad \text{obs. comp.}$

\[ = \ln \frac{1}{z} + y z^{-1} + \frac{y^2}{2} z^{-2} + \frac{y^3}{3} z^{-3} + \cdots \quad \text{abs. conv.} \quad \text{for } |z| > |y| \]

is multipole expansion, prove above them.

Please truncation error $E_p(z) := \ln \frac{1}{z} + \frac{y^p}{p!} z^{-n} - \ln \frac{1}{z-y} = \sum_{n=p}^{\infty} \frac{y^n}{n!} z^{-n}$

\[ \text{p-term approx.} \]

so $|E_p(z)| \leq \sum_{n=p}^{\infty} \frac{|y|^n}{n!} \leq \sum_{n=p}^{\infty} \frac{|y|^n}{n!} \frac{|y|^p}{p!} \leq \frac{|y|^p}{p!} \prod_{n=0}^{p-1} \left( 1 - \frac{n}{p} \right) \quad \text{tail of sum.}$

so $|E_p(z)| = O]\left( \frac{|y|^p}{p!} \right)$ as $p \to \infty$, exponential conv. in $p$.

Use this trick for each charge in disc $B$, get:

Thus (multipole): potential due to $N$ charges $x_j$, locations $y_j$, in disc $B$, $R$, can be rep. by $p$-order multiple expansion in $|z| > b > R$ w/ pointwise error $\leq C \left( \frac{\|y\|}{p!} \|x\| \right)$.
Use to apply off-diagonal block A: \( x_j = b, y_j \sim R \), i.e.,

\[ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} b \\ 1 \\ \vdots \\ 1 \end{bmatrix} \]

Decide \( p \) based on desired accuracy: 

\[ (\text{rough guess}) \approx \epsilon \ll 1, \text{ e.g., } \epsilon = 2^{-20} \text{ to } 2^{-40}. \]

Recipe (HHT):

1. Compute multipole basis due to sources:

\[ C_0 = \sum_{j=1}^{N} X_j, \quad C_n = \sum_{j=1}^{N} \frac{Y_j}{r_j} X_j, \quad n = 1, \ldots, p-1 \]

2. Evaluate multipole expansion at targets:

\[ U(x_i) = \sum_{n=0}^{p-1} C_n x_i^{-n} \]

3. Update the expansion:

\[ U(x_i) = \sum_{n=0}^{p-1} C_n x_i^{-n} \]

Complexity:

- \( O(pN) \)
- \( O(pN) \) total, but \( p = O(\ln \frac{1}{\epsilon}) \)
- \( O(N) \) for fixed error.

Compare original \( O(N^2) \), e.g., \( N = 10^6 \), \( p = 10^{-5} \), \( \epsilon = 10^{-9} \). Then speedup is \( 10^{-15} \approx 30,000 \). This is a good algorithm.

Unfortunately, applying whole of \( A \), interaction matrix between \( \{y_j\}_{j=1}^{N} \), is trickier since not all clumps of points well-separated.

Say all \( y_j \) lie in rectangle, roughly uniformly distributed: 

Cover by \( M \) square boxes:

- \[ \frac{N}{M} \text{ charges per box.} \]
- Effort to get multipole coefficients: \( \sum_{B} \), \( z \to 0 \), replace \( z \) in expansion, where \( z \) is box center.

\[ \sum_{\text{B in box for which B is well-sep.}} \] 

Approx by sum of multipole expansions from each of \( M - q \) other boxes:

Effort \( \propto p(M-q) = O(pM) \).

Choose \( M \) to scale with \( N \) so best?

\[ M = N^2, \text{ balance } \frac{N^L}{N^2} \approx N^2 \]

Fixing \( p \), this is \( O \left( \frac{N^{2+2}}{p} \right) \) times faster than naive, e.g., \( N = 10^6 \), \( p = 10^{-5} \), is \( 10^{-15} \approx 30,000 \).
Last time: \( O(N^2) \) alg. for applying potential matrix \( \Phi \) between \( N \) particles.  

Assumption: uniformly distributed in square box.

Tweak: make \( x \) nearby; sum directly.

Relies on \( A \) matrix coming from elliptic PDE.

Why is \( x \rightarrow Ax \) important to apply \( \Phi \)?  
- enable iterative soln. (“Krylov” methods: apply \( A \) repeatedly) of large Nyström for BVP.
- other apps: compute forces in large gravitational, fluid (vorticity), molecular (electrostatic) sim: methods are either iterative or direct.

Don’t want too many iters (needed for ill-cond.)

\( O(N^3) \) algos: dense solvng Gaussian Elim.

Speed: direct solvers — Gillam’s colly, Hüt Thm.  

Fixed effort, even for ill-cond.

Bottleneck: each target box (N) targets at which target (box) wants \( N \) dipole exps to be evaluated.

Better: combine \( \Phi \) expansions before evaluating at targets in box.

\( \rightarrow \) 2/28/11 (see next page).

Hierarchical (multilevel) versions:

Tree-code:

- small box: \( O(\log N) \) particles.

Once get one box, \( \Phi \) is same.

Gather in groups of \( \log N \) boxes, do \( \Phi \) for \( \log N \) boxes.

Then, \( \Phi \) is \( O(1) \).

Adaptivity:

- what if \( N \) not uniform?  

- Subdivide to different levels until \( O(1) \) changes per box.

- More levels, harder to code.

Fast-multipole method (FMM):

- gather \( \Phi \) at root of tree.

- \( O(N) \) effort close to linear in \( N \).

- For \( N \) of them, separate local \( \Phi \)s as \( \log N \) steps; then do \( \Phi \) on top.

Effort: \( O(N) \), Greengard-Kahllin ‘87.  

Also adaptive version.  

3rd is much faster!
Where the bottleneck? If could make smaller box size L, less interaction could be done directly. But currently would have more boxes hence more effort evaluating all their dipole exps at the (N) distant pts!

Need a way to combine n-pole exps so all target pts in a box can be evaluated from single

expansion... a 'local expansion' = Taylor expansion.

Say z₀ is source box center, rep. by n-pole. One can be rep. by Taylor in z₀.

Consider times in n-pole,

\[ \ln \frac{1}{z_0} = \ln \frac{1}{z_0} - \ln \left( \frac{z}{z_0} - 1 \right) \]

\[ = \ln \frac{1}{z_0} - \frac{z}{z_0} - \frac{1}{2} \frac{z^2}{z_0^2} - \frac{1}{3} \frac{z^3}{z_0^3} - \ldots \]

for box L.

Nth-order \( (z_0)^{-n} \) has

0th Taylor coeff \( = (z_0)^{-n} \int_{z_0} \frac{1}{z_0} \)

1st \( = \frac{1}{m} \int_{z_0} \frac{d^m}{dz^m} (z-z_0)^{-n} \)

\[ = n (z_0)^{-n-1} \]

mth \( = \frac{1}{m} \int_{z_0} \frac{d^m}{dz^m} \ln (z-z_0)^{-n} \)

\[ = n (n+1) \ldots (n+m-1) z_0^{-n-m} \]

So,

Thm: (M2L, "multipole to local") \( u(x) = c_0 \ln \frac{1}{z_0} + \sum_{n=0}^{\infty} c_n (z-z_0)^{-n} \) can be

written as Taylor expansion \[ \sum_n a_n z^n \] abs. convergent in \( |z| < |z_0| \) with coeff.

\[ \left\{ \begin{array}{l}
 c_0 = c_0 \ln \frac{1}{z_0} + \sum_{n=0}^{\infty} c_n n!
 a_n = c_0 \frac{(-1)^n}{n} \sum_{n=1}^{\infty} (-1)^n \frac{n! n}{n} z_0^{-m-C_n} , m = 1, 2, \ldots
 \end{array} \right. \]

Thm: (Error of M2L): if sources \( \sum_{j=1}^{N} z_j \) lie in \( |z-z_0| \leq R \), \( |z_0| > b+R \), for some \( b > R \),

then error of truncation above some to \( p \) terms is, in \( |z| < R \), bounded by \( c \sum_{j=1}^{N} \left| z_j \right| \frac{R^p}{b^p} \)

Ps.: Greengard-Rokhlin '87

Same exponential convergence rate as before:

For focal, can now become:

for each target box compute \( c_{m,n} \) coeff due to each n-pole source box \( c_{m,n} \)

\[ \text{evaluate local (Taylor) exp at all targets in the target box.} \]

\[ \text{Effort is } O(p^2 M^2) \text{ since } p^2 \text{ to map } c_{m,n} \text{ to } a_{m,n} \text{; } L^4 \text{ translation to many one} \]

\[ + O(p N) \text{ eval. } p^th \text{-order local exp. at all } N \text{ target pts.} \]

Total effort now \( p N + q N^2 + p^2 M^2 + p N \) balance, \( M = N \)

\[ \frac{\text{source-to-multipole direct \( M2L \);} \quad \text{let } N = 2 \]  

Overall scaling \( O(p^2 N^{3/2}) \)

\[ \text{if fixed } p \text{. Best yet. Can do even better w/ hierarchical version: FMM.} \]
Recall Helmholtz BVP (eg. for scattering): need N points for \( N \approx yS \) 
- cost; but \( y \) is \( \log \) singular.
- on ring:
- periodic CG; core: linear.
- \( \text{PTIR fails} \).

Are 'cheap' ways to correct \( \text{PTIR} \)? Sagar-Rokhlin: \( ^{37} \) change weights near sing, set diag inj to 0, not 1 SG, \( \approx 60 \) nodes per wavelength needed for high acc.

But is 'product quadrature' \( \approx \text{PTIR} \)? Kowal '91: \( \approx 60 \) nodes per wavelength gets your 14 digits! \( \approx \frac{3}{2} \frac{y}{\pi} \).

\[
\int_0^{2\pi} f(s) \, g(s) \, ds = \sum_{j=1}^{N} w_j f(s_j) \quad \text{where \( w_j = 2 \pi i^{-1} \text{ for PTIR} \).}
\]

\[
\int \text{desired line} \quad \int \text{fixed weight} \quad \int \text{modified, not all \( \frac{2\pi}{N} \) for our \( g \).}
\]

Givens \( g \), how get \( w_{j-1} \) - lets assume \( N \) even (and similar).

\[
\frac{2\pi i}{N} \int_0^{2\pi} f(s) \, e^{ins} \, ds = \sum_{n=-N}^{N} \hat{f}_n \, g_n \int_{-\pi}^{\pi} e^{-ins} \, ds 
= \frac{2\pi i}{N} \hat{f} \, g_n \quad \text{Parseval.}
\]

\[
\frac{2\pi i}{N} \int_0^{2\pi} f(s) \, e^{ins} \, ds \quad \text{a PIA basis.}
\]

if \( f \) smooth, \( \int |f| \to 0 \) fast as \( \frac{1}{m^2} \).

In particular, if \( f \) analytic in strip \( |s| \leq \alpha \), then \( f_n = O(e^{-\alpha |n|}) \) ex: prove this.

Use \( \text{PTIR} \) to approx coeff from \( g_n \): \( f_n = \frac{1}{2\pi i} \int_0^{2\pi} f(s) \, e^{ins} \, ds = 2 \frac{N}{2\pi i} \sum_{j=1}^{N} j^{-1} \hat{f}_j \, g_n \).

Why good? Sub. F. series for \( F \): \( f_n = \frac{1}{2\pi} \sum_{j=1}^{N} j^{-1} \hat{f}_j \, g_n \).

so \( f_n = f_n + \sum_{j=n+1}^{N} f_n + \sum_{m=1}^{n-1} f_m \frac{N}{2\pi} j^{-1} \hat{f}_j \).

so \( \sum_{n=1}^{N} f_n = \sum_{j=1}^{N} \hat{f}_j \).

Then \( \int_0^{2\pi} f(s) \, g(s) \, ds = \frac{\pi}{N} \sum_{n=-N}^{N} \hat{f}_n \, g_n \approx \frac{2\pi}{N} \sum_{n=-N}^{N} \hat{f}_n \, g_n \).

since \( f_n \) exp small for \( |n| > \frac{N}{2} \).

\[
\text{so } w_j = \frac{2\pi i}{N} \hat{f}_j \, g_n \, e^{-i\pi j} \to e^{-\pi |j|}, \text{ i.e. } w_j \approx \text{sinc} |j| \quad \text{PTIR of first } N \text{ Fournier coeff of } f.
\]

Eg. periodized by \( \text{sinc} \): \( f(s) = \ln(4 \sin^2 \frac{s}{2}) \) has \( g_n = \{ 0, n = 0 \text{ or } \mid n \text{ otherwise} \} \)

\[\text{note: } g_n \approx \frac{n}{\sinh n} \quad n \approx 0 \text{, } g \in L^2(\{0,2\pi\})\]
Thus $W_j = \frac{2\pi}{N} \left[ \sum_{n=0}^{M-1} g_{n+j} e^{jn \frac{2\pi}{N}} + g_{n-j} e^{-jn \frac{2\pi}{N}} + \frac{1}{2} (g_{n+ij} e^{ij \frac{2\pi}{N}} + g_{n-ij} e^{-ij \frac{2\pi}{N}}) \right]$

true for any real $j$.

For our $j$, $W_j = \frac{2\pi}{N} \left[ -\sum_{n=1}^{M-1} \frac{\partial}{\partial n} \cos n \pi x_j - (-1)^j \frac{1}{N} \right]$

done.

Note: the $\pi$ in above are exact for $f \in \text{Span} \{ e^{jns} \}_{n=-M}^{M}$, can check.

Now can split kernel of $D - y \bar{S}$ into analytic parts: $\log (\frac{4 - \sin^2 \frac{s - t}{2}}{2})$ analytic, since $\text{new weight}$

shifted cyclically.

$y \bar{S}$ has kernel (vol, parametric $0 \leq s < \pi$):

$\frac{1}{\pi} \int \cos (k |y(s) - y(t)|) |y(s)| \frac{d}{ds} \frac{M_0(kl)}{M_0(\omega_0)} \text{analyt.}$

$M_0(\omega_0)$ defined by $M$, apart from $M(s, \omega_0)$, for which exists formula. the log part precisely

recovered.

Similar for $D$... see Kreiss '81, or Cotter-Kees '88.

Scheme: Nystrom iterate $A_{ij} = \frac{2\pi}{N} M_2(s_i, s_j) + W_{i-j} M_1(s_i, s_j)$

Note: incompatible with FMM, since weights depend on $j$ and so cannot be treated as free change.

Other objects you should see:

A) Sobolev spaces (type of Hilbert spaces)

recall: $L^2(0, 2\pi)$ := $\{ f : f \text{ function } \int_{0}^{2\pi} |f|^2 dx < \infty \}$, loosely

Defn: (Sobolev space order $s$)

$H^s(0, 2\pi) := \{ f : \left( \sum_{n \in \mathbb{Z}} (1 + n^2)^s |f_n|^2 \right)^{\frac{1}{2}} < \infty \}$

$H^0 = L^2$, $s > 0$ enforces faster decay of Fourier coeffs. => smoother than $L^2$.

eg $g(x) = \ln (A \sin^2 \frac{x}{2}) \in H^5(0, 2\pi) \quad \forall s \leq \frac{1}{2}$ since $\sum_{n \in \mathbb{Z}} (1 + n^2)^s |f_n|^2 < \infty$

Thus: Let $s \geq \frac{1}{2}$, $f \in H^s$, then $f \in C[0, 2\pi]$ & periodic

PF: for each $g_n, (\sum_{n \in \mathbb{Z}} |f_n e^{inx}|^2) = \frac{1}{\pi} \int_{0}^{2\pi} |f(x)|^2 dx < \infty$ converges for $s \geq \frac{1}{2}$.
Thm: Let $f \in H^1$, then $\frac{df}{dx} \in H^{-1}$ \quad \text{derivative is less smooth.}

pf: Fourier coeffs of $f$ are in $L^2$.

So $H^1(a,b) = \{ f : \int_a^b |f(x)|^2 \, dx + \int_a^b |f'(x)|^2 \, dx < \infty \}$
\[ \text{since} \quad 2\pi \sum (1 + n^2) |c_n|^2 \]

Thm: single-layer op $S$ is bounded from $H^s$ to $H^{s+1}$

pf sketch: i) singularity of $S$ is $g(s-t) = \ln(4\pi n^{2s-t})$
\[ \text{which has} \quad |g_n| \sim \frac{1}{|n|} \]
\[ \text{as convolution kernels.} \]

ii) Applying convolution op $h(x) = \int g(x-y) y \, dy$ $\Rightarrow$ $h_n = f_n g_n$ in Fourier space.
\[ \text{(check it!)} \]

Thm: $T: H^s \to H^{s-1}$ bounded \quad \text{(order 1)}
\[ T: H^s \to H^{s-1} \text{ bounded}, \quad i.e. \text{like derivative of } 1 \text{ order.} \]

Suggest that $TS$ is order 0. \quad True: $TS = -\frac{i}{4} + (D^s)^2$
\[ \text{called Calderón identity; numerically common to precondition nasty } T \text{ to fast$T$.} \]

\text{Calderón projection (Helmholtz case):} \quad \text{Recall a projection op. $P$ obeys $P^2 = P$ as operators.}\]

Recall interior $GRF: -D\vec{u} + \vec{S}\vec{u} = \vec{u}_\infty \in \mathbb{R}^d$ \n
\[ \text{taking } x \to \mathbb{R}^d \backslash \{ \text{values} \} \quad -\left(\frac{D}{2} + (D^s)^2 \right) \vec{u} = \vec{0} \quad \text{in } \mathbb{R}^d \]
\[ \text{using Fredholm} \quad \vec{u} \quad \text{problem:} \quad -T\vec{u} + (D^s)^2 \vec{u} = \vec{u}_\infty \]
\[ \text{ie} \quad \left[ \begin{array}{c} D - S \\ \frac{1}{T} - D^s \end{array} \right] \left[ \begin{array}{c} \vec{u} \\ \vec{u}_\infty \end{array} \right] = \left[ \begin{array}{c} \vec{0} \\ \vec{0} \end{array} \right] \]
\[ \text{so } P = \left( \begin{array}{c} D^s \end{array} \right) \text{ is identity in } \text{lin. subspace } \mathbb{V}_\infty \text{ of interior bdry data pairs.} \]
Showing $P$ is actually a projection:

$V \rightarrow \mathcal{S}$, know $\mathcal{D}P + \mathcal{S} \mathcal{G}$ is an interior Helmholtz solution, say $u$, in which case $[u_n] \in \mathcal{V}$.

so $P^2 [\xi] = P \left( P [\xi] \right) = \left( \begin{array}{c} u^+ \\ u_n^- \end{array} \right) = \left( \begin{array}{c} u^+ \\ -u_n^- \end{array} \right) = \left( \begin{array}{c} u^+ \\ u_n^- \end{array} \right) = \left( \begin{array}{c} -\xi \\ \eta \end{array} \right)$ True $V(\xi, \eta)$, so $P^2 = P$ as obs. 0.

Since $P_\perp = \frac{1}{2} - H_x = (\frac{1}{2} - H_x)^2 = \frac{1}{4} H_x + H_x^2 = \frac{1}{2} H_x$ so $H_x = \frac{1}{4}$, i.e $\left( \begin{array}{cc} 3/4 & 1/2 \\ 1/2 & -1/4 \end{array} \right) \left( \begin{array}{c} \xi \\ \eta \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$

Thus (exterior GRF): let $(\Delta - k^2)u = 0$ in $\mathbb{R}^4 \setminus \mathcal{S}$ & $u$ sat. radiation condition $\mathcal{O}$, then 

$- \Delta u + \sum u_n = 0 \quad \text{in} \mathbb{R}^4 \setminus \mathcal{S}$

[CK book, Thm. 2.4]

Proof: Let $B$ be ball centered at $0 < \text{radius} < r$ inside $\mathcal{S}$.

We first show $\int_{\partial B} ud\nu = 0(1)$ as $r \rightarrow 0$, i.e.

we have the identity, by expanding, $\int_{\partial B} \left( \frac{\partial u}{\partial r} - ik u \right) d\nu = \int_{\partial B} \left( \frac{\partial u}{\partial r} \right)^2 + k^2 u^2 + 2k \text{Im} \left( \frac{\partial u}{\partial r} \right) d\nu$.

Also, in any region $R$ in which $u$ a Helmholtz soln, we have "flow balance" (FB):

$\text{Im} \int_R \bar{u} \nabla u \, dx \quad \text{by GTT.}$

Integrate by parts into $R$, $= 0$ since $u$ is purely real.

Apply FB to $R = B \setminus \mathcal{S}$ gives

$2k \text{Im} \int_{\partial B} \left( \frac{\partial u}{\partial r} - ik u \right) d\nu = 0$ since $R$ purely real.

Combining (4) gives $\lim_{r \rightarrow 0} \int_{\partial B} \left( \frac{\partial u}{\partial r} \right)^2 + k^2 u^2 d\nu = - F + \lim_{r \rightarrow 0} \int_{\partial B} \left( \frac{\partial u}{\partial r} \right)^2 - ik u^2 d\nu = 0$ by rad. cond.

i) Now use this to show surface term in GRF on $\partial B$ vanishes as $r \rightarrow 0$.

Let $x \in \partial B$, $\mathcal{S}$

$\int_{\partial B} \left( u_n(y) \frac{\partial u(y)}{\partial n} - u_n(y) \Phi(y) \right) ds = \int_{\partial B} \frac{\partial u}{\partial n} - ik u \int_{\partial B} \Phi(u_n - ik u) ds$

Claim $I_1, I_2 \rightarrow 0$ as $r \rightarrow 0$: 

$\cdots$
\[
\frac{\partial \Phi(x,y)}{\partial y} - ik \Phi(x,y) = o\left(\frac{1}{r^\alpha}\right) \quad \text{since } \Phi(x,) \text{ radiating soln.}
\]

by C.S. \[ I_1^2 \leq \int_{\partial B} |u|^2 ds \cdot \int_{\partial B} \left| \Phi^* \Phi \right|^2 ds_y = o(1) \text{ as } r \to \infty. \]

for \( I_2, \quad \Phi(x,) = O\left(\frac{1}{r^\alpha}\right) \) & \( u \) radiating, so \( I_2 \to 0 \) as \( r \to \infty \).

ii) We apply interior GRF to \( B \setminus \delta B \), we get

\[ \int_{\partial B} u(x,y) \Phi(x,y) - u(y) \frac{\partial \Phi(x,y)}{\partial y} ds_y = \left\{ \begin{array}{ll}
0 & \text{if } x \in \Omega \\
\text{True for each } r. \text{ Finally take lim } r \to \infty. \end{array} \right. \]

Q.E.D.

May now finish Calderon Projectors:

apply exterior GRF, take \( x \to 2\delta \) & use TR's given, \( P_+ \left[ u_{x+} \right] = \left[ u_{x+} \right], \quad P_- \left[ u_{x+} \right] = \left[ 0 \right] \)

for any radiative Helm. soln. \( u \in L^*(S) \), by identical proof as \( P_- \), we then get \( P_+ \) is a projection.

Lemma: \( \mathbf{V}_+ \oplus \mathbf{V}_- = \mathbf{L}^2(\Omega) \quad \text{Pf: } I = I_2 + H = P_+ + P_- \)

so, \( \forall \Phi \in \mathbf{V}, \left[ \Phi \right] = P_+ \left[ \Phi \right] + P_- \left[ \Phi \right] \), is a decomposition into \( \mathbf{V}_+ \& \mathbf{V}_- \).

Summary: \( P_+ \mathbf{V}_+ = \mathbf{V}_+ \), \( P_+ \mathbf{V}_- = \{0\} \quad \text{and} \quad P_- \mathbf{V}_+ = \{0\}, \quad P_- \mathbf{V}_- = \mathbf{V}_- \)

Thus \( P_+, P_- \) are complementary projectors.

Note: We haven't shown \( \mathbf{V}_+ \perp \mathbf{V}_- \), i.e. these projections are orthogonal. This would require \( P_+ P_- = P_+ P_- = 0 \)

Shipman Venakides paper e.g. 2003, have clear explanation of this.

We haven't shown \( \mathbf{V}_+ \perp \mathbf{V}_- \), i.e. these projections are orthogonal. This would require \( P_+ P_- = P_+ P_- = 0 \), etc; I don't believe holds.