Then w is a bounded, harmonic function on $\mathbb{R}^n$. Then, by Liouville's Theorem, w must be constant. Therefore, we conclude that

$$\begin{aligned} \widetilde{u}(x) &= u(x) + C \\ &= \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy + C, \end{aligned}$$

as claimed.

□

3.3 Solving Laplace's Equation on Bounded Domains

3.3.1 Laplace's Equation on a Rectangle

In this section, we will solve Laplace's equation on a rectangle in $\mathbb{R}^2$. First, we consider the case of Dirichlet boundary conditions. That is, we consider the following boundary value problem. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\}$. We want to look for a solution of the following,

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega \\ u(0,y) = g_1(y),\ u(a,y) = g_2(y) & 0 < y < b \\ u(x,0) = g_3(x),\ u(x,b) = g_4(y) & 0 < x < a. \end{cases} \tag{3.5}$$

In order to do so, we consider the following simpler example. From this, we will show how to solve the more general problem above.

Example 10. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\}$. Consider

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega \\ u(0,y) = g_1(y),\ u(a,y) = 0 & 0 < y < b \\ u(x,0) = 0,\ u(x,b) = 0 & 0 < x < a. \end{cases} \tag{3.6}$$

We use separation of variables. We look for a solution of the form

$$u(x,y) = X(x)Y(y).$$

Plugging this into our equation, we get

$$X''Y + XY'' = 0.$$

Now dividing by XY, we arrive at

$$\frac{X''}{X} + \frac{Y''}{Y} = 0,$$

which implies

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda$$

for some constant λ. By our boundary conditions, we want $Y(0) = 0 = Y(b)$. Therefore, we begin by solving the eigenvalue problem,

$$\begin{cases} Y'' = -\lambda Y & 0 < y < b \\ Y(0) = 0 = Y(b). \end{cases}$$

As we know, the solutions of this eigenvalue problem are given by

$$Y_n(y) = \sin\left(\frac{n\pi}{b}y\right), \ \lambda_n = \left(\frac{n\pi}{b}\right)^2.$$

We now turn to solving

$$X'' = \left(\frac{n\pi}{b}\right)^2 X$$

with the boundary condition $X(a) = 0$. The solutions of this ODE are given by

$$X_n(x) = A_n \cosh\left(\frac{n\pi}{b}x\right) + B_n \sinh\left(\frac{n\pi}{b}x\right).$$

Now the boundary condition $X(a) = 0$ implies

$$A_n \cosh\left(\frac{n\pi}{b}a\right) + B_n \sinh\left(\frac{n\pi}{b}a\right) = 0.$$

Therefore,

$$u_n(x, y) = X_n(x)Y_n(y) = \left[A_n \cosh\left(\frac{n\pi}{b}x\right) + B_n \sinh\left(\frac{n\pi}{b}x\right)\right] \sin\left(\frac{n\pi}{b}y\right)$$

where A_n, B_n satisfy the condition

$$A_n \cosh\left(\frac{n\pi}{b}a\right) + B_n \sinh\left(\frac{n\pi}{b}a\right) = 0.$$

is a solution of Laplace's equation on Ω which satisfies the boundary conditions $u(x, 0) = 0$, $u(x, b) = 0$, and $u(a, y) = 0$. As we know, Laplace's equation is linear. Therefore, we can take any combination of solutions $\{u_n\}$ and get a solution of Laplace's equation which satisfies these three boundary conditions. Therefore, we look for a solution of the form

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} \left[A_n \cosh\left(\frac{n\pi}{b}x\right) + B_n \sinh\left(\frac{n\pi}{b}x\right)\right] \sin\left(\frac{n\pi}{b}y\right)$$

where A_n, B_n satisfy

$$A_n \cosh\left(\frac{n\pi}{b}a\right) + B_n \sinh\left(\frac{n\pi}{b}a\right) = 0. \tag{3.7}$$

To solve our boundary-value problem (3.6), it remains to find coefficients A_n, B_n which not only satisfy (3.7), but also satisfy the condition $u(0, y) = g_1(y)$. That is, we need

$$u(0, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{b}y\right) = g_1(y).$$

That is, we want to be able to express g_1 in terms of its Fourier sine series on the interval $[0, b]$. Assuming g_1 is a "nice" function, we can do this. From our earlier discussion of Fourier series, we know that the Fourier sine series of a function g_1 is given by

$$g_1(y) \sim \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{b} y\right)$$

where the coefficients A_n are given by

$$A_n = \frac{\langle g_1, \sin\left(\frac{n\pi}{b} y\right)\rangle}{\langle \sin\left(\frac{n\pi}{b} y\right), \sin\left(\frac{n\pi}{b} y\right)}$$

where the L^2-inner product is taken over the interval $[0, b]$.

Therefore, to summarize, we have found a solution of (3.6) given by

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} \left[A_n \cosh\left(\frac{n\pi}{b} x\right) + B_n \sinh\left(\frac{n\pi}{b} x\right)\right] \sin\left(\frac{n\pi}{b} y\right)}$$

where

$$\boxed{A_n = \frac{\langle g_1, \sin\left(\frac{n\pi}{b} y\right)\rangle}{\langle \sin\left(\frac{n\pi}{b} y\right), \sin\left(\frac{n\pi}{b} y\right)}}$$

and

$$\boxed{B_n = -\coth\left(\frac{n\pi}{b} a\right) A_n.}$$

$\diamond$

Now we return to considering (3.5). For the general boundary value problem on a rectangle with Dirichlet boundary conditions, we can find a solution by finding four separate solutions u_i for $i = 1, \ldots, 4$ such that each u_i is identically zero on three of the sides and satisfies the boundary condition on the fourth side. For example, for the boundary value problem (3.5), we use the procedure in the above example to find a function $u_1(x, y)$ which is harmonic on Ω and such that $u_1(0, y) = g_1(y)$ and $u_1(a, y) = 0$ for $0 < y < b$, and $u_1(x, 0) = 0 = u_1(x, b)$ for $0 < x < a$. Similar we find functions u_2, u_3 and u_4 which vanish on three of the sides but satisfy the fourth boundary condition.

We now consider an example where we have a mixed boundary condition on one side.

Example 11. Let $\Omega = \{(x, y) \in \mathbb{R}^2, 0 < x < L, 0 < y < H\}$. Consider the following boundary value problem,

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x, y) \in \Omega \\ u(0, y) = 0, \ u(L, y) = 0 & 0 < y < H \\ u(x, 0) - u_y(x, 0) = 0, \ u(x, H) = f(x) & 0 < x < L. \end{cases} \tag{3.8}$$

Using separation of variables, we have

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$