NOTES FOR 128: COMBINATORIAL REPRESENTATION THEORY OF COMPLEX LIE ALGEBRAS AND RELATED TOPICS

(FIRST COUPLE LECTURES – MORE ONLINE AS WE GO)

Recommended reading

[Bou] N. Bourbaki, *Elements of Mathematics: Lie Groups and Algebras.*
Not always easy to read from front to back, but it was clearly written by the oracles of mathematics at the time, with the purpose of containing everything.

[FH] W. Fulton, J. Harris, *Representation Theory: A first course.*
Written for the non-specialist, but rich with examples and pictures. Mostly, an example-driven tour of finite-dimensional representations of finite groups and Lie algebras and groups. Cheap – buy this book.

[Hum] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory.*
Lightweight approach to finite-dimensional Lie algebras. Has a lot of the proofs, but not a lot of examples.

[Ser] J. J. Serre, *Complex Semisimple Lie Algebras.*
Super lightweight. A tour of the facts, without much proof, but great quick reference.

The logistics

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Website: http://www.math.dartmouth.edu/∼m128s14/

Pre-reqs: Graduate algebra or equivalent.

Homework: Complete a minimum of five assignments (out of at least 10).

Office hours: By appointment, but check the website's calendar for drop-in.

Class cancellation: There will also be a conference at CRM in Montreal on Combinatorial Representation Theory, April 21–25. Class will be canceled that Wednesday and Friday. You can either come with me, or complete a small project, to be assigned later.

1. The poster child of CRT: the symmetric group

Combinatorial representation theory (CRT) is the study of representations of algebraic objects, using combinatorics to keep track of the relevant information. To see what I mean, let's take a look at the symmetric group.

Let F be your favorite field of characteristic 0. Recall that an *algebra* A over F is a vector space over F with an associative multiplication

$$A \otimes A \to A$$

Here, the tensor product is over F, and just means that the multiplication is bilinear. Our favorite examples for a while will be

(1) Group algebras (today)
(2) Enveloping algebras of Lie algebras (tomorrow-ish)

And our favorite field is $F = \mathbb{C}$.

The *symmetric group* S_k is the group of permutations of $\{1, \dots, k\}$. The *group algebra* $\mathbb{C}S_k$ is the vector space

$$\mathbb{C}S_k = \left\{ \sum_{\sigma \in S_k} c_\sigma \sigma \mid c_\sigma \in \mathbb{C} \right\}$$

with multiplication linear and associative by definition:

$$\left(\sum_{\sigma \in S_k} c_\sigma \sigma \right) \left(\sum_{\pi \in S_k} d_\pi \pi \right) = \sum_{\sigma, \pi \in G} (c_\sigma d_\pi)(\sigma \pi).$$

Example. When $k = 3$,

$$S_3 = \{1, (12), (23), (123), (132), (13)\} = \langle s_1 = (12), s_2 = (23) \mid s_1^2 = s_2^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle.$$

So

$$\mathbb{C}S_3 = \{c_1 + c_2(12) + c_3(23) + c_4(123) + c_5(132) + c_6(13) \mid c_i \in \mathbb{C}\}$$

and, for example,

$$\begin{aligned}(2 + (12))(5(123) - (23)) &= 10(123) - 2(23) + 5(12)(123) - (12)(23) \\ &= 10(123) - 2(23) + 5(23) - (123) = \boxed{3(23) + 9(123)}.\end{aligned}$$

1.1. **Our best chance of understanding big bad algebraic structures: representations!** A *homomorphism* is a structure-preserving map. A *representation* of an F-algebra A is a vector space V over F, together with a homomorphism

$$\rho : A \to \mathrm{End}(V) = \{ \text{ } F\text{-linear maps } V \to V \text{ } \}.$$

The map (equipped with the vector space) is the representation; the vector space (equipped with the map) is called an *A-module*.

Example. Favorite representation of S_n is the permutation representation: Let $V = \mathbb{C}^k = \mathbb{C}\{v_1, \dots, v_k\}$. Define

$$\rho : S_k \to \mathrm{GL}_k(\mathbb{C}) \qquad \text{by} \qquad \rho(\sigma) v_i = v_{\sigma(i)}$$

$k = 2$:

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rho(\mathbb{C}S_2) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{C} \right\} \subset \mathrm{End}(\mathbb{C}^2)$$

$k = 3$:

$$1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (132) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad (13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\rho(\mathbb{C}S_3) = \left\{ \begin{pmatrix} a+c & b+e & d+f \\ b+d & a+f & c+e \\ e+f & c+d & a+b \end{pmatrix} \;\middle|\; a,b,c,d,e,f \in \mathbb{C} \right\} \subset \mathrm{End}(\mathbb{C}^3)$$

A representation/module V is *simple* or *irreducible* if V has no invariant subspaces.

Example. The permutation representation is not simple since $v_1 + \cdots + v_k = (1, \ldots, 1)$ is invariant, and so $T = \mathbb{C}\{(1, \ldots, 1)\}$ is a submodule (called the *trivial representation*). However, the trivial representation is one-dimensional, and so is clearly simple. Also, the orthogonal compliment of T, given by

$$S = \mathbb{C}\{v_2 - v_1, v_3 - v_1, \ldots, v_k - v_1\}$$

is also simple (called the *standard representation*). So V *decomposes* as

$$V = T \oplus S \tag{1.1}$$

by the change of basis

$$\{v_1, \ldots, v_k\} \to \{v, w_2, \ldots, w_k\} \qquad \text{where } v = v_1 + \cdots + v_k \text{ and } w_i = v_i - v_1.$$

New representation looks like

$$\rho(\sigma)v = v, \qquad \rho(\sigma)w_i = w_{\sigma(i)} - w_{\sigma(1)} \qquad \text{where } w_1 = 0.$$

For example, when $k = 3$,

$$1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (132) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix} \qquad (13) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

Notice, the vector space $\mathrm{End}(\mathbb{C}^2)$ is four-dimensional, and the four matrices

$$\rho_S(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho_S((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\rho_S((23)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \rho_S((13)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

are linearly independent, so $\rho_S(\mathbb{C}S_3) = \mathrm{End}(\mathbb{C}^2)$, and so (at least for $k = 3$) S is also simple! So the decomposition in (1.1) is complete.

An algebra is *semisimple* if all of its modules decompose into the sum of simple modules.

Example. The group algebra of a group G over a field F is semisimple iff $\mathrm{char}(F)$ does not divide $|G|$. So group algebras over $\mathbb{C}$ are all semisimple.

We like semisimple algebras because they are isomorphic to a direct sum of the matrix rings of their simple modules (*Artin-Wedderburn theorem*). So studying a semisimple algebra is "the same" as studying its simple modules.

Theorem 1.1. *For a finite group G, the irreducible representations of G are in bijection with its conjugacy classes.*

Proof.

(A) Show

(1) the class sums of G, given by

$$\left\{ \sum_{h \in \mathcal{K}} h \mid \mathcal{K} \text{ is a conjugacy class of } G \right\}$$

form a basis for $Z(FG)$;
Example: $G = S_3$. The class sums are

$$1, \quad (12) + (23) + (13), \quad \text{and} \quad (123) + (132)$$

(2) and $\dim(Z(FG)) = |\hat{G}|$ where $\hat{G}$ is an indexing set of the irreducible representations of G.

(B) Use character theory. A *character* χ of a group G corresponding to a representation ρ is a homomorphism

$$\chi_\rho : G \to \mathbb{C} \qquad \text{defined by} \qquad \chi_\rho : g \to \mathrm{tr}(\rho(g)).$$

Nice facts about characters:

(1) They're *class functions* since

$$\chi_\rho(hgh^{-1}) = \mathrm{tr}(\rho(hgh^{-1})) = \mathrm{tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \mathrm{tr}(\rho(g)) = \chi_\rho(g).$$

Example. The character associated to the trivial representation of any group G is $\chi_1 = 1$.

Example. Let χ be the character associate to the standard representation of S_3. Then

$$\chi(1) = 2, \qquad \chi((12)) = \chi((23)) = \chi((13)) = 0, \qquad \chi((123)) = \chi(132) = -1.$$

(2) They satisfy nice relations like

$$\chi_{\rho\oplus\psi} = \chi_\rho + \chi_\psi$$
$$\chi_{\rho\otimes\psi} = \chi_\rho \chi_\psi$$

(3) The characters associated to the irreducible representations form an orthonormal basis for the class functions on G. (This gives the bijection)

Studying the representation theory of a group is "the same" as studying the character theory of that group.

This is not a particularly satisfying bijection, either way. It doesn't say "given representation X, here's conjugacy class Y, and vice versa." $\square$

Conjugacy classes of the symmetric group are given by cycle type. For example the conjugacy classes of S_4 are

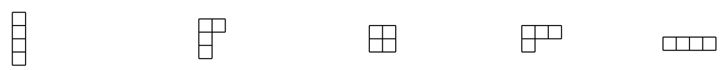
$$\begin{aligned}
&\{1\} = \{(a)(b)(c)(d)\}\\
&\{(12),(13),(14),(23),(24),(34)\} = \{(ab)(c)(d)\}\\
&\{(12)(34),(13)(24),(14)(23)\} = \{(ab)(cd)\}\\
&\{(123),(124),(132),(134),(142),(143),(234),(243)\} = \{(abc)(d)\}\\
&\{(1234),(1243),(1324),(1342),(1423),(1432)\} = \{(abcd)\}.
\end{aligned}$$

Cycle types of permutations of k are in bijection with *partitions* $\lambda \vdash k$:

$$\lambda = (\lambda_1, \lambda_2, \dots) \quad \text{with } \lambda_1 \geq \lambda_2 \geq \dots, \quad \lambda_i \in \mathbb{Z}_{\geq 0}, \lambda_1 + \lambda_2 + \cdots = k.$$

The cycle types and their corresponding partitions of 4 are

$(a)(b)(c)(d)$	$(ab)(c)(d)$	$(ab)(cd)$	$(abc)(d)$	$(abcd)$
$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)



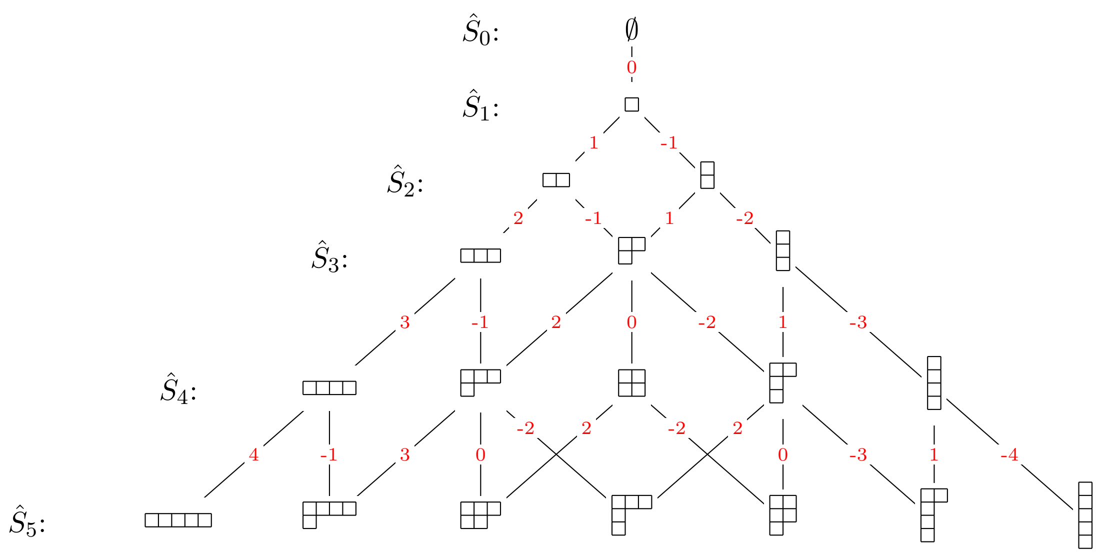
where the picture is an up-left justified arrangement of boxes with λ_i boxes in the ith row.

The combinatorics goes way deep!

Young's Lattice:

Vertices: Label vertices in label vertices on level k with partitions of k.

Edges: Draw and edge from a partition of k to a partition of $k+1$ if they differ by a box.



Some combinatorial facts: (without proof)

(1) The representations of S_k are indexed by the partitions on level k.

(2) The basis for the module corresponding to a partition λ is indexed by downward-moving paths from $\emptyset$ to λ.

(3) The representation is encoded combinatorially as well. Define the *content* of a box b in row i and column j of a partition as

$$c(b) = j - i, \qquad \text{the diagonal number of } b.$$

Label each edge in the diagram by the content of the box added. The matrix entries for the transposition $(i\ i+1)$ are functions of the values on the edges between levels $i-1$, i, and $i+1$.

(4) If S^λ is the module indexed by λ, then

$$\mathrm{Ind}_{S_k}^{S_{k+1}}(S^\lambda) = \bigoplus_{\substack{\mu \vdash k+1 \\ \lambda - \mu}} S^\mu \qquad \text{and } \mathrm{Res}_{S_{k-1}}^{S_k}(S^\lambda) = \bigoplus_{\substack{\mu \vdash k-1 \\ \mu - \lambda}} S^\mu$$

1.2. **Where is this all going?** Really, where has this all gone? The symmetric group is so nice in so many ways, that we've chased these combinatorial features down many paths.

One path is the study of other reflection groups, both finite and not. That took us to their deformations, called Hecke algebras, and other spin-off Hecke-like algebras and diagram algebras.

Another came from Schur-Weyl duality, which showed that the representation theory of S_k as k ranges, is in duality with the representation theory of $\mathrm{GL}_n(\mathbb{C})$. Then, later, people got into Lie algebras, and saw that the same results held there, and that combinatorics controls most of complex Lie theory as well. Further, there are lots of important deformations of Lie algebras whose combinatorics is also controlled combinatorially.

References

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[BR] H. Barcelo, A. Ram, *Combinatorial Representation Theory,* New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97), 23–90, Math. Sci. Res. Inst. Publ., 38, Cambridge Univ. Press, Cambridge, 1999.

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