**Answers to Exercise 1:** Some things about the classical Lie algebras.

(1) For each of the following types, give a basis $B$ which has exactly $r$ diagonal matrices and is otherwise as workable as possible. In particular, keep symmetry, so that if $x \in B$ then $x^T \in B$. Express elements as sums of elementary matrices $E_{ij}$ (the matrix with a 1 in the $(i, j)$ position and 0's elsewhere. Clearly, we're choosing a basis for $V$ in the process. A form on $V$ defined by a matrix $J$ is defined by $\langle u, v\rangle = u^T J v$. Let $I_r$ be the $r \times r$ identity matrix.

(a) **Type $A_r$.** For $r \geq 1$, give a basis for $\mathfrak{sl}_{r+1}$, and verify that $\dim(\mathfrak{sl}_{r+1}) = r(r + 2)$.

A good basis of $\mathfrak{sl}_{r+1}$ is

$$\{E_{ii} - E_{i+1,i+1} \mid i = 1, \ldots, r\} \sqcup \{E_{ij}, E_{ij} \mid 1 \leq i < j \leq r + 1\}.$$

(b) **Type $C_r$.** For $r \geq 1$, put the form on $V = \mathbb{C}^{2r}$ given by $J = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$.

($*$) Verify that $\langle , \rangle$ is skew symmetric, i.e. $\langle u, v\rangle = -\langle v, u\rangle$.

Since $J^T = -J$,

$$\langle u, v\rangle = u^T J v = (v^T J^T u)^T = -(v^T J u)^T = -\langle v, u\rangle^T = -\langle v, u\rangle.$$

($*$) Verify that if $\mathfrak{sp}_{2r} = \{x \in \mathfrak{sl}(V) \mid \langle xu, v\rangle = -\langle u, xv\rangle\}$, then $\mathfrak{sp}_{2r}$ is in fact closed ($\langle , \rangle$ is bilinear, so you only need to check $[,]$.)

For any bilinear form $\langle , \rangle$ on $\mathbb{C}^n$, any subspace of $\mathfrak{gl}_n$ given by

$$\mathfrak{s} = \{x \in \mathfrak{gl}_n \mid \langle xu, v\rangle = \langle v, xu\rangle\}$$

is a Lie algebra since (1) it's a subspace because $\langle , \rangle$ is bilinear, and (2) it's closed under the Lie bracket because

$$\begin{aligned} \langle [x, y]u, v\rangle &= \langle (xy - yx)u, v\rangle = \langle xyu, v\rangle - \langle yxu, v\rangle \\ &= -\langle yu, xv\rangle + \langle xu, yv\rangle = \langle u, yxv\rangle - \langle u, xyv\rangle \\ &= -\langle u, (xy - yx)v\rangle = -\langle u, [x, y]v\rangle. \end{aligned}$$

($*$) Give a basis for $\mathfrak{sp}_{2r}$, and verify that $\dim(\mathfrak{sp}_{2r}) = r(2r + 1)$. (Break each $x \in \mathfrak{sp}_{2r}$ into the four $r \times r$ matrices that $J$ effect independently, (see below) and get conditions on each of them)

The condition $x^T J = -Jx$ requires $X^T = -Z$, $Y^T = Y$, and $(Y')^T = Y'$. A good basis for $\mathfrak{sp}_{2r}$ is

$$\{E_{ij} - E_{j+r,i+r} \mid 1 \leq i, j \leq r\} \sqcup \{E_{i,r+j} + E_{j,r+i}, E_{r+i,j} + E_{r+j,i} \mid 1 \leq i \leq j \leq r\}.$$

(c) **Type $D_r$.** For $r \geq 2$, put the form on $V = \mathbb{C}^{2r}$ given by $J = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}$.

($*$) Verify that $\langle , \rangle$ is symmetric, i.e. $\langle u, v\rangle = \langle v, u\rangle$.

Since $J^T = J$, a similar computation as in (b) will show $\langle u, v\rangle = \langle v, u\rangle$.

(∗) Verify that if $\mathfrak{so}_{2r} = \{x \in \mathfrak{sl}(V) \mid \langle xu, v\rangle = -\langle u, xv\rangle\}$, then $\mathfrak{so}_{2r}$ is closed.

See part (b).

(∗) Give a basis for $\mathfrak{so}_{2r}$, and verify that $\dim(\mathfrak{so}_{2r}) = r(2r-1)$. (Break each $x \in \mathfrak{so}_{2r}$ into the four $r \times r$ matrices that $J$ effects independently, (see below) and get conditions on each of them)

The condition $x^T J = -Jx$ requires $X^T = -Z$, $Y^T = -Y$, and $(Y')^T = -Y'$. A good basis for $\mathfrak{so}_{2r}$ is

$$\{E_{ij} - E_{j+r,i+r} \mid 1 \le i, j \le r\} \sqcup \{E_{i,r+j} - E_{j,r+i}, E_{r+i,j} - E_{r+j,i} \mid 1 \le i < j \le r\}.$$

(d) **Type $B_r$.** For $r \ge 1$, put the form on $V = \mathbb{C}^{2r+1}$ given by $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_r \\ 0 & I_r & 0 \end{pmatrix}$. Give a basis for $\mathfrak{so}_{2r+1}$, and verify that $\dim(\mathfrak{so}_{2r+1}) = r(2r+1)$. (Break each $x \in \mathfrak{so}_{2r+1}$ into the nine blocks that $J$ effects independently (see below) and get conditions on each of them.)

The condition $x^T J = -Jx$ requires $X^T = -Z$, $Y^T = -Y$, and $(Y')^T = -Y'$, $a = 0$, $b^T = -c'$, and $c^T = -b'$. A good basis for $\mathfrak{so}_{2r+1}$ is

$$\begin{aligned} &\{E_{i+1,j+1} - E_{j+1+r,i+1+r} \mid 1 \le i, j \le r\} \\ &\qquad \sqcup \{E_{i+1,r+j+1} - E_{j+1,r+i+1}, E_{r+i+1,j+1} - E_{r+j+1,i+1} \mid 1 \le i < j \le r\} \\ &\qquad \sqcup \{E_{1,r+i+1} - E_{i+1,1}, E_{1,i+1} - E_{r+i+1,1}\}. \end{aligned}$$

(2) As mentioned in class, $B_1$, $C_1$, $C_2$, $D_1$, $D_2$, and $D_3$ are either not distinct from, or decompose into direct sums of Lie algebras from amongst.

$$\{A_r\}_{r\ge1} \sqcup \{B_r\}_{r\ge2} \sqcup \{C_r\}_{r\ge3} \sqcup \{D_r\}_{r\ge4}$$

Verify this for any 4 of these 6 Lie algebras by expressing them in terms of the others.

$$B_1 \cong C_1 \cong A_1, \quad D_1 \cong \mathbb{C},$$
$$D_2 \cong A_1 \times A_1, \quad C_2 \cong B_2, \quad D_3 \cong A_3.$$

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Decompositions of elements for $\mathfrak{g}$ of each type.