Exercise 1: Some things about the classical Lie algebras.

(1) For each of the following types, give a basis B which has exactly r diagonal matrices and is otherwise as workable as possible. In particular, keep symmetry, so that if $x \in B$ then $x^T \in B$. Express elements as sums of elementary matrices E_{ij} (the matrix with a 1 in the (i, j) position and 0's elsewhere. Clearly, we're choosing a basis for V in the process. A form on V defined by a matrix J is defined by $\langle u, v\rangle = u^T Jv$. Let I_r be the $r \times r$ identity matrix.

(a) **Type A_r.** For $r \geq 1$, give a basis for $\mathfrak{sl}_{r+1}$, and verify that $\dim(\mathfrak{sl}_{r+1}) = r(r + 2)$.

(b) **Type C_r.** For $r \geq 1$, put the form on $V = \mathbb{C}^{2r}$ given by $J = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$.

(∗) Verify that $\langle , \rangle$ is skew symmetric, i.e. $\langle u, v\rangle = -\langle v, u\rangle$.

(∗) Verify that if $\mathfrak{sp}_{2r} = \{x \in \mathfrak{sl}(V) \mid \langle xu, v\rangle = -\langle u, xv\rangle\}$, then $\mathfrak{sp}_{2r}$ is in fact closed ($\langle , \rangle$ is bilinear, so you only need to check $[,]$.)

(∗) Give a basis for $\mathfrak{sp}_{2r}$, and verify that $\dim(\mathfrak{sp}_{2r}) = r(2r + 1)$. (Break each $x \in \mathfrak{sp}_{2r}$ into the four $r \times r$ matrices that J effect independently, (see below) and get conditions on each of them)

(c) **Type D_r.** For $r \geq 2$, put the form on $V = \mathbb{C}^{2r}$ given by $J = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}$.

(∗) Verify that $\langle , \rangle$ is symmetric, i.e. $\langle u, v\rangle = \langle v, u\rangle$.

(∗) Verify that if $\mathfrak{so}_{2r} = \{x \in \mathfrak{sl}(V) \mid \langle xu, v\rangle = -\langle u, xv\rangle\}$, then $\mathfrak{so}_{2r}$ is closed.

(∗) Give a basis for $\mathfrak{so}_{2r}$, and verify that $\dim(\mathfrak{so}_{2r}) = r(2r - 1)$. (Break each $x \in \mathfrak{so}_{2r}$ into the four $r \times r$ matrices that J effects independently, (see below) and get conditions on each of them)

(d) **Type B_r.** For $r \geq 1$, put the form on $V = \mathbb{C}^{2r+1}$ given by $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_r \\ 0 & I_r & 0 \end{pmatrix}$. Give a basis for $\mathfrak{so}_{2r+1}$, and verify that $\dim(\mathfrak{so}_{2r+1}) = r(2r + 1)$. (Break each $x \in \mathfrak{so}_{2r+1}$ into the nine blocks that J effects independently (see below) and get conditions on each of them.)

(2) As mentioned in class, B_1, C_1, C_2, D_1, D_2, and D_3 are either not distinct from, or decompose into direct sums of Lie algebras from amongst.

$$\{A_r\}_{r\geq 1} \sqcup \{B_r\}_{r\geq 2} \sqcup \{C_r\}_{r\geq 3} \sqcup \{D_r\}_{r\geq 4}$$

Verify this for any 4 of these 6 Lie algebras by expressing them in terms of the others.

Decompositions of elements for $\mathfrak{g}$ of each type.