Exercise 1: Some things about the classical Lie algebras.

(1) For each of the following types, give a basis $B$ which has exactly $r$ diagonal matrices and is otherwise as workable as possible. In particular, keep symmetry, so that if $x \in B$ then $x^T \in B$. Express elements as sums of elementary matrices $E_{ij}$ (the matrix with a 1 in the $(i,j)$ position and 0’s elsewhere. Clearly, we’re choosing a basis for $V$ in the process. A form on $V$ defined by a matrix $J$ is defined by $\langle u, v \rangle = u^T J v$. Let $I_r$ be the $r \times r$ identity matrix.

(a) **Type $A_r$.** For $r \geq 1$, give a basis for $\mathfrak{sl}_{r+1}$, and verify that $\dim(\mathfrak{sl}_{r+1}) = r(r + 2)$.

(b) **Type $C_r$.** For $r \geq 1$, put the form on $V = \mathbb{C}^{2r}$ given by $J = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$.

   (*) Verify that $\langle \cdot, \cdot \rangle$ is skew symmetric, i.e. $\langle u, v \rangle = -\langle v, u \rangle$.

   (*) Verify that if $\mathfrak{sp}_{2r} = \{ x \in \mathfrak{sl}(V) \mid \langle xu, v \rangle = -\langle u, xv \rangle \}$, then $\mathfrak{sp}_{2r}$ is in fact closed (\langle \cdot, \cdot \rangle is bilinear, so you only need to check $[,]$.)

   (*) Give a basis for $\mathfrak{sp}_{2r}$, and verify that $\dim(\mathfrak{sp}_{2r}) = r(2r + 1)$. (Break each $x \in \mathfrak{sp}_{2r}$ into the four $r \times r$ matrices that $J$ effect independently, (see below) and get conditions on each of them)

(c) **Type $D_r$.** For $r \geq 2$, put the form on $V = \mathbb{C}^{2r}$ given by $J = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}$.

   (*) Verify that $\langle \cdot, \cdot \rangle$ is symmetric, i.e. $\langle u, v \rangle = \langle v, u \rangle$.

   (*) Verify that if $\mathfrak{so}_{2r} = \{ x \in \mathfrak{sl}(V) \mid \langle xu, v \rangle = -\langle u, xv \rangle \}$, then $\mathfrak{so}_{2r}$ is closed.

   (*) Give a basis for $\mathfrak{so}_{2r}$, and verify that $\dim(\mathfrak{so}_{2r}) = r(2r - 1)$. (Break each $x \in \mathfrak{so}_{2r}$ into the four $r \times r$ matrices that $J$ effects independently, (see below) and get conditions on each of them)

(d) **Type $B_r$.** For $r \geq 1$, put the form on $V = \mathbb{C}^{2r+1}$ given by $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_r \\ 0 & I_r & 0 \end{pmatrix}$. Give a basis for $\mathfrak{so}_{2r+1}$, and verify that $\dim(\mathfrak{so}_{2r+1}) = r(2r + 1)$. (Break each $x \in \mathfrak{so}_{2r+1}$ into the nine blocks that $J$ effect independently (see below) and get conditions on each of them.)

(2) As mentioned in class, $B_1, C_1, C_2, D_1, D_2$, and $D_3$ are either not distinct from, or decompose into direct sums of Lie algebras from amongst:

\[ \{ A_r \}_{r \geq 1} \sqcup \{ B_r \}_{r \geq 2} \sqcup \{ C_r \}_{r \geq 3} \sqcup \{ D_r \}_{r \geq 4} \]

Verify this for any 4 of these 6 Lie algebras by expressing them in terms of the others.

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Decompositions of elements for $\mathfrak{g}$ of each type.

- **$C_r, D_r$**
  - $X, Y$
  - $Y', Z$

- **$B_r$**
  - $a$
  - $b$
  - $c$
  - $X, Y$
  - $Y', Z$

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