Answers to Exercise 2: Some things about $\mathfrak{sl}_2$.
Recall, $L(d)$ is the irreducible $\mathfrak{sl}_2$-module with dimension $d+1$.

(1) Calculate (give the matrices for) the adjoint representation of $\mathfrak{sl}_2$ and decompose it into irreducible summands.

On the basis $\{x, h, y\}$,

$$\mathrm{ad}_x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \mathrm{ad}_y = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}.$$

Since the action of h is diagonalized, we can read the weights directly off of the diagonal terms in ad_h, i.e. the weights of the adjoint representation are $\{2, 0, -2\}$ with multiplicity one. So the adjoint representation is isomorphic to $L(2)$.

(2) Let $V = \{u, v\}$ be the standard representation of $\mathfrak{sl}_2$, given by

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Classify V as an $\mathfrak{sl}_2$-module (as sums of $L(d)$'s).

Since the action of h is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the weights of V are $\{1, -1\}$ with multiplicity one. Thus $V \cong L(1)$.

(b) Give a basis for $V \otimes V$ and calculate the matrices for the action of x, y, and h on that basis (For $g \in \mathfrak{sl}_2$, g acts on $a \otimes b$ by...).

$\mathfrak{sl}_2$ on V	u	v
x	0	u
y	v	0
h	1	-1

$\mathfrak{sl}_2$ on $V \otimes V$	$u \otimes u$	$u \otimes v$	$v \otimes u$	$v \otimes v$
x	0	$u \otimes u$	$u \otimes u$	$u \otimes v + v \otimes u$
y	$v \otimes u + u \otimes v$	$v \otimes v$	$v \otimes v$	0
h	$2(u \otimes u)$	0	0	$2(v \otimes v)$

(c) Classify $V \otimes V$ and $V^{\otimes 3}$ as $\mathfrak{sl}_2$-modules.

Since $V = L(1)$, as we saw in class, $V \otimes V \cong L(2) \oplus L(0)$ and $V \otimes V \otimes V \cong L(3) \oplus L(0)^{\oplus 2}$.

(d) (Bonus) Provide a general formula for the decomposition of $V^{\otimes k}$.

(3) With V as in the previous part, define the kth symmetric sum of V as

$$\mathrm{Sym}^k(V) = V^{\otimes k} / \langle a \otimes b - b \otimes a \rangle \cong \mathbb{C}\{u^k, u^{k-1}v, \dots, v^k\}$$

(since $\mathrm{Sym}^k(V)$ is isomorphic to the degree-k homogeneous elements of $\mathbb{C}[u, v]$).

(a) Generally describe the action of $\mathfrak{sl}_2$ on $\mathrm{Sym}^k(V)$. (For $g \in \mathfrak{sl}_2$, g acts on $u^\ell v^{k-\ell}$ by...)

For example, when $k = 2$, the action table is gotten from the action table from the previous part modded out by commutativity, i.e.

$\mathfrak{sl}_2$ on $\mathrm{Sym}^2(V)$	u^2	$uv = vu$	v^2
x	0	u^2	$2uv$
y	$2uv$	v^2	0
h	$2u^2$	0	$2v^2$

In general,

$$x \cdot (u^\ell v^{k-\ell}) = (k-\ell)(u^{\ell+1} v^{k-\ell-1}), \quad y \cdot (u^\ell v^{k-\ell}) = \ell(u^{\ell-1} v^{k-\ell+1}),$$
$$\text{and} \quad h \cdot (u^\ell v^{k-\ell}) = (2\ell - k)(u^\ell v^{k-\ell}).$$

(b) How does $\mathrm{Sym}^k(V)$ decompose into irreducible summands?

Since $u^k, u^{k-1}v, \ldots, v^k$ are weight vectors with weights $k, k-2, \ldots, -k$, respectively, $\mathrm{Sym}^k(V) \cong L(k)$.

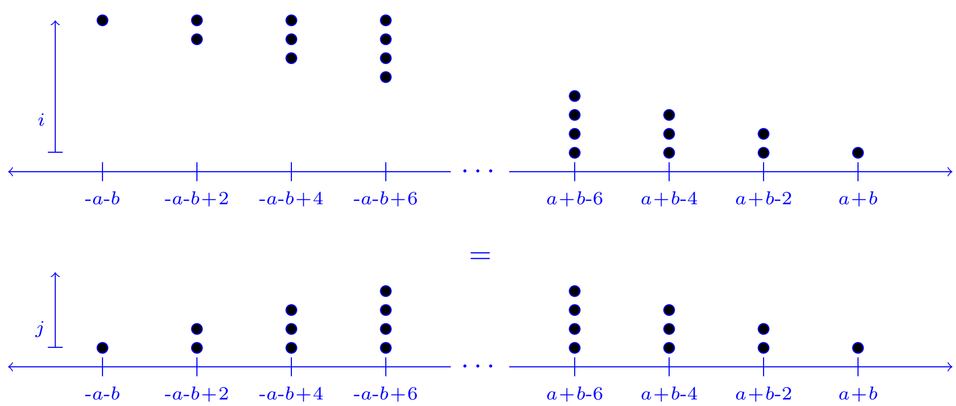
(c) Show that for $a \geq b$,

$$\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V) \cong \bigoplus_{i=0}^{b} \mathrm{Sym}^{a+b-2i}(V).$$

Since the weight spaces of $M \otimes N$ are given by $(M \otimes N)_\gamma = \bigoplus_{\alpha+\beta=\gamma} M_\alpha \otimes N_\beta$, the weights of $L(a) \otimes L(b)$ are

$$\bigsqcup_{i=0}^{b} \{a+b-2i, a-2+b-2i, \ldots, -a+b-2i\} \qquad \text{(counting multiplicities)}$$
$$= \bigsqcup_{j=0}^{b} \{a+b-2i, a+b-2i-2, \ldots, -(a+b-2i)\}$$
$$= \bigsqcup_{j=0}^{b} \{\text{weights of } L(a+b-2i)\}.$$

Pictorially,



So

$$\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V) \cong L(a) \otimes L(b) \cong \bigoplus_{i=0}^{b} L(a+b-2i) \cong \bigoplus_{i=0}^{b} \mathrm{Sym}^{a+b-2i}(V).$$