Exercise 2: Some things about $\mathfrak{sl}_2$.
Recall, $L(d)$ is the irreducible $\mathfrak{sl}_2$-module with dimension $d+1$.

(1) Calculate (give the matrices for) the adjoint representation of $\mathfrak{sl}_2$ and decompose it into irreducible summands.

(2) Let $V = \{u, v\}$ be the standard representation of $\mathfrak{sl}_2$, given by

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  (a) Classify V as an $\mathfrak{sl}_2$-module (as sums of $L(d)$'s).

  (b) Give a basis for $V \otimes V$ and calculate the matrices for the action of x, y, and h on that basis (For $g \in \mathfrak{sl}_2$, g acts on $a \otimes b$ by...).

  (c) Classify $V \otimes V$ and $V^{\otimes 3}$ as $\mathfrak{sl}_2$-modules.

  (d) (Bonus) Provide a general formula for the decomposition of $V^{\otimes k}$.

(3) With V as in the previous part, define the kth symmetric sum of V as

$$\mathrm{Sym}^k(V) = V^{\otimes k} / \langle a \otimes b - b \otimes a \rangle \cong \mathbb{C}\{u^k, u^{k-1}v, \dots, v^k\}$$

(since $\mathrm{Sym}^k(V)$ is isomorphic to the degree-k homogeneous elements of $\mathbb{C}[u, v]$).

  (a) Generally describe the action of $\mathfrak{sl}_2$ on $\mathrm{Sym}^k(V)$. (For $g \in \mathfrak{sl}_2$, g acts on $u^\ell v^{k-\ell}$ by...)

  (b) How does $\mathrm{Sym}^k(V)$ decompose into irreducible summands?

  (c) Show that for $a \geq b$,

$$\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V) \cong \bigoplus_{i=0}^{b} \mathrm{Sym}^{a+b-2i}(V).$$