Exercise 3: Some things about NIBS forms.

(1) Prove that the Killing form is an invariant symmetric bilinear form on any simple finite dimensional complex Lie algebra.

The Killing form $(\cdot, \cdot) : g \otimes g \to \mathbb{C}$ is given by $(x, y) = \text{Tr}(\text{ad}x \text{ad}y)$. It’s symmetric since $\text{Tr}(AB) = \text{Tr}(BA)$. It’s linear in the first coordinate because $\text{ad}$ and trace are both linear, so

$$\langle ax + by, z \rangle = \text{Tr}(\text{ad}_{ax+by} \text{ad}z) = \text{Tr}((\text{ad}x + \text{ad}y) \text{ad}z) \quad = a \text{Tr}(\text{ad}_x \text{ad}z) + b \text{Tr}(\text{ad}_y \text{ad}z) = a \langle x, z \rangle + b \langle y, z \rangle.$$  

But $(\cdot, \cdot)$ is symmetric, so it’s bilinear. Since $\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$, it’s invariant because

$$\langle \text{ad}_x(y), z \rangle = \langle [x, y], z \rangle = \text{Tr}((\text{ad}_x \text{ad}_y)[z]) = \text{Tr}((\text{ad}_x \text{ad}_y - \text{ad}_y \text{ad}_x)[z]) \quad = \text{Tr}((\text{ad}_y \text{ad}_x - \text{ad}_x \text{ad}_y)[z]) \quad = \langle y, [z, x] \rangle = -\langle y, \text{ad}_x(z) \rangle.$$  

(2) Show that the trace form on the standard representation of $\mathfrak{sl}_n$ is non-degenerate.

We just need to check that for every element of the basis $B = \{E_{i,j}, E_{\ell,\ell} - E_{\ell+1,\ell+1} \mid 1 \leq i \neq j \leq n, \ell = 1, \ldots, n-1\}$ has some other element of $\mathfrak{sl}_n$ with which it pairs non-trivially. Indeed,

$$\langle E_{i,j}, E_{j,i} \rangle = \text{Tr}(E_{i,j} E_{j,i}) = \text{Tr}(E_{i,i}) = 1$$

and

$$\langle E_{\ell,\ell} - E_{\ell+1,\ell+1}, E_{\ell,\ell} - E_{\ell+1,\ell+1} \rangle = \text{Tr}((E_{\ell,\ell} - E_{\ell+1,\ell+1})^2) = \text{Tr}(E_{\ell,\ell} + E_{\ell+1,\ell+1}) = 2.$$  

(3) Pick two of the classical types $(A_r, B_r, C_r, D_r)$ and calculate how the trace form on the standard representation of each type differs from the Killing form (as a function of $r$). (You’ll need a good basis for each to do this.)

If $g$ is simple, then any NIBS form is a scalar of the Killing form. So we only need to calculate one pairing in each form and take the quotient.

Type $A_r$. We saw in class how to use the fact that

$$(a, b)_{\text{ad}} = \sum_{\alpha \in R} \alpha(a)\alpha(b) \quad \text{for all } a, b \in \mathfrak{h}$$

to quickly calculate, say, $\langle h_1, h_1 \rangle_{\text{ad}}$ using the roots of $\mathfrak{sl}_{r+1}$. Here’s another slightly less slick, but totally straightforward calculation of the same constant ratio.

For the trace form on the standard representation $\text{st}$, $\langle E_{1,2}, E_{2,1} \rangle_{\text{st}} = 1$.

For the Killing form, we need to calculate $\text{ad}_{E_{1,2}}$ and $\text{ad}_{E_{2,1}}$. One basis is

$$\{E_{i,j}, h_\ell = E_{\ell,\ell} - E_{\ell+1,\ell+1} \mid 1 \leq i \neq j \leq r + 1, 1 \leq \ell \leq r \}.$$  

Then for any $i,j$,

$$\text{ad}_{E_{1,2}}(E_{i,j}) = \delta_{i,2} E_{1,j} - \delta_{j,1} E_{i,2} \quad \text{and} \quad \text{ad}_{E_{2,1}}(E_{i,j}) = \delta_{i,1} E_{2,j} - \delta_{j,2} E_{i,1}.$$
So
\[ \text{ad}_{E_1,2} \text{ad}_{E_2,1} E_{i,j} = \text{ad}_{E_1,2} (\delta_{i,1} E_{2,j} - \delta_{j,2} E_{i,1}) \]
\[ = \delta_{i,1} (E_{1,j} - \delta_{j,1} E_{2,2}) - \delta_{j,2} (\delta_{i,2} E_{1,1} - E_{i,2}) \]
\[ = \delta_{i,1} E_{1,j} + \delta_{j,2} E_{i,2} - (\delta_{i,1} \delta_{j,1} E_{2,2} + \delta_{j,2} \delta_{i,2} E_{1,1}). \]

Thus
\[ \text{ad}_{E_1,2} \text{ad}_{E_2,1} E_{i,j} = \delta_{i,1} E_{i,j} + \delta_{j,2} E_{i,j} \quad \text{for } i \neq j, \]
and
\[ \text{ad}_{E_1,2} \text{ad}_{E_2,1} (E_{\ell,\ell} - E_{\ell+1,\ell+1}) = \begin{cases} E_{1,1} - E_{2,2} - E_{2,2} + E_{1,1} = 2h_1 & \ell = 1 \\ E_{2,2} - E_{1,1} = -h_1 & \ell = 2 \\ 0 & \text{otherwise}. \end{cases} \]

So the trace of \( \text{ad}_{E_1,2} \text{ad}_{E_2,1} \), which is the sum over basis elements \( b \) of the coefficient of \( b \) in \( \text{ad}_{E_1,2} \text{ad}_{E_2,1} b \), is given by
\[ \frac{r}{\#E_{1,1}} + \frac{r}{\#E_{2,2}} + \frac{2}{h_1} = 2(r + 1) \]
\[ (E_{1,2} \text{ gets double counted, but } \text{ad}_{E_1,2} \text{ad}_{E_2,1} E_{1,2} = 2E_{1,2}). \]

So
\[ \langle \cdot \rangle_{\text{ad}} = 2(r + 1) \langle \cdot \rangle_{\text{st}}. \]

Other types. Let \( \text{st} \) be the standard representation.

Type \( B_r \): \( \langle \cdot \rangle_{\text{ad}} = (2r - 1) \langle \cdot \rangle_{\text{st}} \)
Type \( C_r \): \( \langle \cdot \rangle_{\text{ad}} = 2(r + 1) \langle \cdot \rangle_{\text{st}} \)
Type \( D_r \): \( \langle \cdot \rangle_{\text{ad}} = 2(r - 1) \langle \cdot \rangle_{\text{st}} \)

(4) Let \( B = \{b_1, \ldots, b_\ell\} \) be a basis for a finite-dimensional reductive complex Lie algebra \( \mathfrak{g} \) with a NIBS form \( \langle \cdot, \cdot \rangle \), and define the dual basis
\[ B^* = \{b_1^*, \ldots, b_\ell^*\} \quad \text{by} \quad \langle b_i, b_j^* \rangle = \delta_{i,j}. \]

The \textit{Casimir} element of \( \mathfrak{g} \) is
\[ \kappa = \sum_{i=1}^\ell b_i b_i^* \in U \mathfrak{g}. \]

Prove the following.
(4) \( \kappa \) does not depend on the choice of basis.

Note first that \( \{b_1^*, \ldots, b_\ell^*\} \) is also a basis of \( \mathfrak{g} \). Let \( \{d_1, \ldots, d_\ell\} \) be a third basis of \( \mathfrak{g} \). Then \( b_i = \sum_j \langle b_i, d_j^* \rangle d_j \) implies
\[ \kappa = \sum_{i=1}^\ell b_i b_i^* = \sum_{i,j=1}^\ell \langle b_i, d_j^* \rangle d_j b_i^* \]
\[ = \sum_{j=1}^\ell d_j \left( \sum_i \langle b_i, d_j^* \rangle b_i^* \right) = \sum_{j=1}^\ell d_j d_j^*. \]
(b) $\kappa \in Z(U\mathfrak{g})$, where $Z(U\mathfrak{g})$ is the center of $U\mathfrak{g}$ (it suffices to show that $\kappa$ commutes with every element of $\mathfrak{g}$).

Let $x \in \mathfrak{g}$. Then

$$x\kappa = \sum_{i=1}^{\ell} xb_i b_i^* = \sum_{i=1}^{\ell} ([x, b_i] + b_i x) b_i^*$$

$$= \sum_{i,j=1}^{\ell} ([x, b_i] b_j^*) b_j b_i^* + \sum_{i=1}^{\ell} b_i x b_i^*$$

$$= - \sum_{i,j=1}^{\ell} (b_i, [x, b_j^*]) b_j b_i^* + \sum_{i=1}^{\ell} b_i x b_i^*$$

$$= - \sum_{j=1}^{\ell} b_j [x, b_j^*] + \sum_{i=1}^{\ell} b_i x b_i^*$$

$$= \sum_{i=1}^{\ell} b_i (-x b_i + b_i x + x b_i) = \kappa x.$$ 

[Notice that (i) $B^*$ is also a basis for $\mathfrak{g}$, and (ii) for any basis $B = \{b_i\}$ and $x \in \mathfrak{g}$, you have $x = \sum_i (x, b_i^*) b_i.$]