Exercise 3: Some things about NIBS forms.

(1) Prove that the Killing form is an invariant symmetric bilinear form on any simple finite dimensional complex Lie algebra.

The Killing form $\langle , \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ is given by $\langle x, y\rangle = \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y)$. It's symmetric since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. It's linear in the first coordinate because ad and trace are both linear, so

$$\begin{aligned}\langle ax + by, z\rangle &= \mathrm{Tr}(\mathrm{ad}_{ax+by}\mathrm{ad}_z) = \mathrm{Tr}((a\mathrm{ad}_x + b\mathrm{ad}_y)\mathrm{ad}_z) = \mathrm{Tr}(a\mathrm{ad}_x\mathrm{ad}_z + b\mathrm{ad}_y\mathrm{ad}_z)\\ &= a\mathrm{Tr}(\mathrm{ad}_x\mathrm{ad}_z) + b(\mathrm{Tr}(\mathrm{ad}_y\mathrm{ad}_z) = a\langle x, z\rangle + b\langle y, z\rangle.\end{aligned}$$

But $\langle , \rangle$ is symmetric, so it's bilinear. Since $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB)$, it's invariant because

$$\begin{aligned}\langle \mathrm{ad}_x(y), z\rangle = \langle [x, y], z\rangle &= \mathrm{Tr}(\mathrm{ad}_{[x,y]}\mathrm{ad}_z) = \mathrm{Tr}((\mathrm{ad}_x\mathrm{ad}_y - \mathrm{ad}_y\mathrm{ad}_x)\mathrm{ad}_z)\\ &= \mathrm{Tr}(\mathrm{ad}_x\mathrm{ad}_y\mathrm{ad}_z) - \mathrm{Tr}(\mathrm{ad}_y\mathrm{ad}_x\mathrm{ad}_z) = \mathrm{Tr}(\mathrm{ad}_y\mathrm{ad}_z\mathrm{ad}_x) - \mathrm{Tr}(\mathrm{ad}_y\mathrm{ad}_x\mathrm{ad}_z)\\ &= \mathrm{Tr}(\mathrm{ad}_y(\mathrm{ad}_z\mathrm{ad}_x - \mathrm{ad}_x\mathrm{ad}_z)) = \mathrm{Tr}(\mathrm{ad}_y\mathrm{ad}_{[z,x]})\\ &= \langle y, [z, x]\rangle = -\langle y, \mathrm{ad}_x(z)\rangle.\end{aligned}$$

(2) Show that the trace form on the standard representation of $\mathfrak{sl}_n$ is non-degenerate.

We just need to check that for every element of the basis $B = \{E_{i,j}, E_{\ell,\ell} - E_{\ell+1,\ell+1} \mid 1 \leq i \neq j \leq n, \ell = 1, \ldots, n-1\}$ has some other element of $\mathfrak{sl}_n$ with which it pairs non-trivially. Indeed,

$$\langle E_{i,j}, E_{j,i}\rangle = \mathrm{Tr}(E_{i,j}E_{j,i}) = \mathrm{Tr}(E_{i,i}) = 1$$

and

$$\langle E_{\ell,\ell} - E_{\ell+1,\ell+1}, E_{\ell,\ell} - E_{\ell+1,\ell+1}\rangle = \mathrm{Tr}((E_{\ell,\ell} - E_{\ell+1,\ell+1})^2) = \mathrm{Tr}(E_{\ell,\ell} + E_{\ell+1,\ell+1}) = 2.$$

(3) Pick two of the classical types (A_r, B_r, C_r, D_r) and calculate how the trace form on the standard representation of each type differs from the Killing form (as a function of r). (You'll need a good basis for each to do this.)

If $\mathfrak{g}$ is simple, then any NIBS form is a scalar of the Killing form. So we only need to calculate one pairing in each form and take the quotient.
Type A_r. We saw in class how to use the fact that

$$\langle a, b\rangle_{\mathrm{ad}} = \sum_{\alpha \in R} \alpha(a)\alpha(b) \qquad \text{for all } a, b \in \mathfrak{h}$$

to quickly calculate, say, $\langle h_1, h_1\rangle_{\mathrm{ad}}$ using the roots of $\mathfrak{sl}_{r+1}$. Here's another slightly less slick, but totally straightforward calculation of the same constant ratio.

For the trace form on the standard representation st, $\langle E_{1,2}, E_{2,1}\rangle_{\mathrm{st}} = 1$.

For the Killing form, we need to calculate $\mathrm{ad}_{E_{1,2}}$ and $\mathrm{ad}_{E_{2,1}}$. One basis is

$$\{E_{i,j}, h_\ell = E_{\ell,\ell} - E_{\ell+1,\ell+1} \mid 1 \leq i \neq j \leq r+1, 1 \leq \ell \leq r\}.$$

Then for any i, j,

$$\mathrm{ad}_{E_{1,2}}(E_{i,j}) = \delta_{i,2}E_{1,j} - \delta_{j,1}E_{i,2} \quad \text{and} \quad \mathrm{ad}_{E_{2,1}}(E_{i,j}) = \delta_{i,1}E_{2,j} - \delta_{j,2}E_{i,1}.$$

So

$$\begin{aligned}\mathrm{ad}_{E_{1,2}}\mathrm{ad}_{E_{2,1}}E_{i,j} &= \mathrm{ad}_{E_{1,2}}(\delta_{i,1}E_{2,j} - \delta_{j,2}E_{i,1}) \\ &\delta_{i,1}(E_{1,j} - \delta_{j,1}E_{2,2}) - \delta_{j,2}(\delta_{i,2}E_{1,1} - E_{i,2}) \\ &= \delta_{i,1}E_{1,j} + \delta_{j,2}E_{i,2} - (\delta_{i,1}\delta_{j,1}E_{2,2} + \delta_{j,2}\delta_{i,2}E_{1,1}).\end{aligned}$$

Thus

$$\mathrm{ad}_{E_{1,2}}\mathrm{ad}_{E_{2,1}}E_{i,j} = \delta_{i,1}E_{i,j} + \delta_{j,2}E_{i,j} \quad \text{for } i \neq j,$$

and

$$\mathrm{ad}_{E_{1,2}}\mathrm{ad}_{E_{2,1}}(E_{\ell,\ell} - E_{\ell+1,\ell+1}) = \begin{cases} E_{1,1} - E_{2,2} - E_{2,2} + E_{1,1} = 2h_1 & \ell = 1 \\ E_{2,2} - E_{1,1} = -h_1 & \ell = 2 \\ 0 & \text{otherwise.}\end{cases}$$

So the trace of $\mathrm{ad}_{E_{1,2}}\mathrm{ad}_{E_{2,1}}$, which is the sum over basis elements b of the coefficient of b in $\mathrm{ad}_{E_{1,2}}\mathrm{ad}_{E_{2,1}}b$, is given by

$$\underbrace{r}_{\#E_{1,j}} + \underbrace{r}_{\#E_{i,2}} + \underbrace{2}_{h_1} = 2(r+1)$$

($E_{1,2}$ gets double counted, but $\mathrm{ad}_{E_{1,2}}\mathrm{ad}_{E_{2,1}}E_{1,2} = 2E_{1,2}$). So

$$\boxed{\langle,\rangle_{\mathrm{ad}} = 2(r+1)\langle,\rangle_{\mathrm{st}}.}$$

Other types. Let st be the standard representation.

$$\begin{aligned}&\text{Type } B_r\text{:} \quad \langle,\rangle_{\mathrm{ad}} = (2r-1)\langle,\rangle_{\mathrm{st}} \\ &\text{Type } C_r\text{:} \quad \langle,\rangle_{\mathrm{ad}} = 2(r+1)\langle,\rangle_{\mathrm{st}} \\ &\text{Type } D_r\text{:} \quad \langle,\rangle_{\mathrm{ad}} = 2(r-1)\langle,\rangle_{\mathrm{st}}\end{aligned}$$

(4) Let $B = \{b_1, \ldots, b_\ell\}$ be a basis for a finite-dimensional reductive complex Lie algebra $\mathfrak{g}$ with a NIBS form $\langle,\rangle$, and define the dual basis

$$B^* = \{b_1^*, \ldots, b_\ell^*\} \quad \text{by} \quad \langle b_i, b_j^*\rangle = \delta_{i,j}.$$

The *Casimir* element of $\mathfrak{g}$ is

$$\kappa = \sum_{i=1}^{\ell} b_i b_i^* \in U\mathfrak{g}.$$

Prove the following.

(a) κ does not depend on the choice of basis.

Note first that $\{b_1^*, \ldots, b_\ell^*\}$ is also a basis of $\mathfrak{g}$. Let $\{d_1, \ldots, d_\ell\}$ be a third basis of $\mathfrak{g}$. Then $b_i = \sum_j \langle b_i, d_j^*\rangle d_j$ implies

$$\begin{aligned}\kappa = \sum_{i=1}^{\ell} b_i b_i^* &= \sum_{i,j=1}^{\ell} \langle b_i, d_j^*\rangle d_j b_i^* \\ &= \sum_{j=1}^{\ell} d_j \left(\sum_i \langle b_i, d_j^*\rangle b_i^*\right) = \sum_{j=1}^{\ell} d_j d_j^*.\end{aligned}$$

(b) $\kappa \in Z(U\mathfrak{g})$, where $Z(U\mathfrak{g})$ is the center of $U\mathfrak{g}$ (it suffices to show that κ commutes with every element of $\mathfrak{g}$).

Let $x \in \mathfrak{g}$. Then

$$\begin{aligned}
x\kappa &= \sum_{i=1}^{\ell} x b_i b_i^* = \sum_{i=1}^{\ell} ([x, b_i] + b_i x) b_i^* \\
&= \sum_{i,j=1}^{\ell} \langle [x, b_i], b_j^* \rangle b_j b_i^* + \sum_{i=1}^{\ell} b_i x b_i^* \\
&= -\sum_{i,j=1}^{\ell} \langle b_i, [x, b_j^*] \rangle b_j b_i^* + \sum_{i=1}^{\ell} b_i x b_i^* \\
&= -\sum_{j=1}^{\ell} b_j [x, b_j^*] + \sum_{i=1}^{\ell} b_i x b_i^* \\
&= \sum_{i=1}^{\ell} b_i (-x b_i + b_i x + x b_i) = \kappa x.
\end{aligned}$$

[Notice that (i) B^* is also a basis for $\mathfrak{g}$, and (ii) for any basis $B = \{b_i\}_i$ and $x \in \mathfrak{g}$, you have $x = \sum_i \langle x, b_i^* \rangle b_i$.]