Exercise 3: Some things about NIBS forms.

(1) Prove that the Killing form is an invariant symmetric bilinear form on any simple finite dimensional complex Lie algebra.

(2) Show that the trace form on the standard representation of $\mathfrak{sl}_n$ is non-degenerate.

(3) Pick two of the classical types (A_r, B_r, C_r, D_r) and calculate how the trace form on the standard representation of each type differs from the Killing form (as a function of r). (You'll need a good basis for each to do this.)

(4) Let $B = \{b_1, \dots, b_\ell\}$ be a basis for a finite-dimensional reductive complex Lie algebra $\mathfrak{g}$ with a NIBS form $\langle , \rangle$, and define the dual basis

$$B^* = \{b_1^*, \dots, b_\ell^*\} \quad \text{by} \quad \langle b_i, b_j^* \rangle = \delta_{i,j}.$$

The *Casimir* element of $\mathfrak{g}$ is

$$\kappa = \sum_{i=1}^{\ell} b_i b_i^* \in U\mathfrak{g}.$$

Prove the following.

(a) κ does not depend on the choice of basis.

(b) $\kappa \in Z(U\mathfrak{g})$, where $Z(U\mathfrak{g})$ is the center of $U\mathfrak{g}$ (it suffices to show that κ commutes with every element of $\mathfrak{g}$).

[Notice that (i) B^* is also a basis for $\mathfrak{g}$, and (ii) for any basis $B = \{b_i\}_i$ and $x \in \mathfrak{g}$, you have $x = \sum_i \langle x, b_i^* \rangle b_i$.]