Exercise 4: Some things about roots.

(1) (a) Calculate the roots for types B_r, C_r, and D_r.

Let

$$\varepsilon_i : h_\ell \to \begin{cases} \mathrm{Tr}(h_\ell E_{i,i}) & \text{in cases } C_r \text{ and } D_r\text{, and} \\ \mathrm{Tr}(h_\ell E_{i+1,i+1}) & \text{in case } B_r, \end{cases}$$

where

$$h_\ell = \begin{cases} E_{\ell,\ell} - E_{\ell+r,\ell+r} & \text{in cases } C_r \text{ and } D_r\text{, and} \\ E_{\ell+1,\ell+1} - E_{\ell+r+1,\ell+r+1} & \text{in case } B_r. \end{cases}$$

So in all cases, $\delta_{i\ell} = \varepsilon_i(\ell)$. Therefore, with respect to the trace form on the standard representation $\langle , \rangle$, $h_{\varepsilon_i} = \frac{1}{2}h_\ell$, and $\{\varepsilon_1, \ldots, \varepsilon_r\}$ is an orthonormal basis for $\mathfrak{h}_{\mathbb{R}}^*$.
The root are as follows.
Type B_r:

$$R = \{\pm\varepsilon_k, \pm(\varepsilon_i - \varepsilon_j), \pm(\varepsilon_i + \varepsilon_j) \mid 1 \le k \le r, 1 \le i < j \le r\}$$

Type C_r:

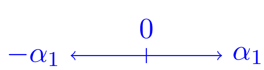
$$R = \{\pm 2\varepsilon_k, \pm(\varepsilon_i - \varepsilon_j), \pm(\varepsilon_i + \varepsilon_j) \mid 1 \le k \le r, 1 \le i < j \le r\}$$

Type D_r:

$$R = \{\pm(\varepsilon_i - \varepsilon_j), \pm(\varepsilon_i + \varepsilon_j) \mid 1 \le i < j \le r\}$$

(b) Draw the roots for B_1, B_2, C_1, C_2, D_1, and D_2 (these can all be drawn in one or two dimensions).
Note: compare your pictures to your answers for Exercise 1, part (2)!

Type B_1, C_1:



where $\alpha_1 = \begin{cases} \varepsilon_1 & \text{type } B_1, \\ 2\varepsilon_1 & \text{type } C_1. \end{cases}$

This is the same picture as for A_1 (where $\alpha = \varepsilon_1 - \varepsilon_2$), and we saw $B_1, C_1 \cong A_1$.
Type D_1 has no roots (so the picture is the 0-dimensional point), and $D_1 \cong \mathbb{C}$.
Type B_2:

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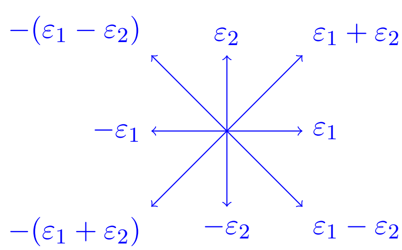
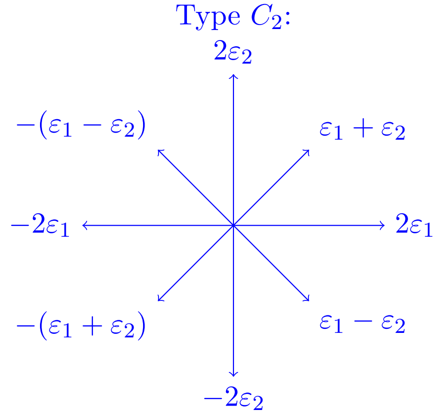
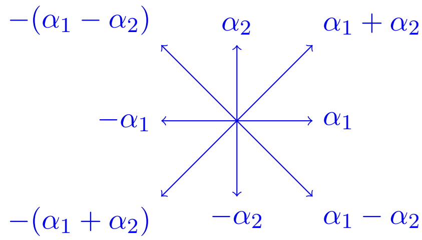


Type C_2:



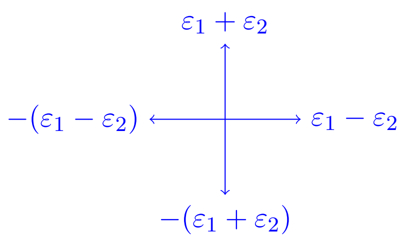
These diagrams are not a priori the same, even though we saw $B_2 \cong C_2$. But by rotating the C_2 roots by $45°$, we see they both look like

Type B_2 and C_2:



where $\alpha_1 = \begin{cases} \varepsilon_1 & \text{in type } B_2, \\ \varepsilon_1 - \varepsilon_2 & \text{in type } C_2, \end{cases}$ and $\alpha_2 = \begin{cases} \varepsilon_2 & \text{in type } B_2, \\ \varepsilon_1 + \varepsilon_2 & \text{in type } C_2. \end{cases}$

Type D_2:



Notice that this looks exactly like the direct product of the roots of A_1 with itself, and indeed $D_2 \cong A_1 \times A_1$.

(2) For $\alpha, \beta \in R$, show that
   (a) $\beta(h_{\alpha^\vee}) \in \mathbb{Z}$,
   (b) $\beta - \beta(h_{\alpha^\vee})\alpha \in R$, and
   (c) if $\beta \neq \pm\alpha$, and a and b are the largest non-negative integers such that

$$\beta - a\alpha \in R \qquad \text{and } \beta + b\alpha \in R,$$

   then $\beta + i\alpha \in R$ for all $-a \leq i \leq b$ and $\beta(h_{\alpha^\vee}) = a - b$.

(Use the fact that $\sum_i \mathfrak{g}_{\beta+i\alpha}$ is a $\mathfrak{sl}_2$-module.)

Note that $\alpha(h_{\alpha^\vee}) = 2$.

Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$,

$$V = \sum_i \mathfrak{g}_{\beta+i\alpha}$$

is a simple $\mathfrak{s}_\alpha$ submodule of $\mathfrak{g}$ (under the adjoint action). The action of $h_{\alpha^\vee}$ on $\mathfrak{g}_{\beta+i\alpha}$ is by the constant

$$\langle \alpha^\vee, \beta + i\alpha \rangle = \langle \alpha^\vee, \beta \rangle + 2i.$$

Since V is a $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$ module, the weights of $h_{\alpha^\vee}$ are all integers of the same parity, and symmetric around 0.

Since $\beta \in R$, $\mathfrak{g}_\beta \neq 0$, so $\langle \alpha^\vee, \beta \rangle$ is a weight of V. So $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$. And since $\langle \alpha^\vee, \beta \rangle$ is a weight of V, so is $-\langle \alpha^\vee, \beta \rangle$. Which $\mathfrak{g}_{\beta+i\alpha}$ has that weight? It's exactly when

$$-\langle \alpha^\vee, \beta \rangle = \langle \alpha^\vee, \beta + i\alpha \rangle = \langle \alpha^\vee, \beta \rangle + 2i.$$

So $i = -\langle\alpha^\vee, \beta\rangle$. Therefore $\mathfrak{g}_{\beta - \langle\alpha^\vee, \beta\rangle\alpha\rangle} \neq 0$, so

$$\beta - \langle\alpha^\vee, \beta\rangle\alpha\rangle = \beta - \beta(h_{\alpha^\vee})\alpha \in R.$$

Finally, if $\beta \neq \pm\alpha$, and a and b are the largest non-negative integers such that

$$\beta - a\alpha \in R \qquad \text{and } \beta + b\alpha \in R,$$

Then one of of each extreme is the highest weight space of V and the other is the lowest, so every integral shift of α between must be a non-zero weight space as well.