Exercise 5: Some things about weights and representations.

(1) Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra.

(a) Show that if $L(\lambda)$ and $L(\mu)$ are highest weight modules (of weights λ and μ), show that $L(\lambda)\otimes L(\mu)$ has $L(\lambda+\mu)$ as a submodule with multiplicity 1. (Think about primitive elements)

Proof. In a highest weight module $L(\mu)$, the spanning set of $\{yv_\mu^+ \mid y = y_1^{m_1}\cdots y_\ell^{m_\ell}, y_i \in \mathfrak{g}_{-\alpha_i}\}$ is a weight spanning set, where the weight of yv_μ^+ is $\mu - \sum_i m_i\alpha_i$. So the set $\{yv_\lambda^+ \otimes y'v_\mu^+ \mid y = y_1^{n_1}\cdots y_\ell^{n_\ell}, y' = y_1^{m_1}\cdots y_\ell^{m_\ell}\}$ is a weight spanning set for $L(\lambda)\otimes L(\mu)$, where the weight of $v_\lambda^+\otimes v_\mu^+$ is

$$(\lambda - \sum_i n_i\alpha_i) + (\mu - \sum_i m_i\alpha_i) = \lambda+\mu-\sum_i (n_i+m_i)\alpha_i.$$

So the multiplicity of the weight space or weight $\lambda+\mu$ is one, and is generated by $v_\lambda^+\otimes v_\mu^+$. Further, $v_\lambda^+\otimes v_\mu^+$ is primitive. So $L(\lambda+\mu)$ is a submodule of $L(\lambda)\otimes L(\mu)$ with multiplicity 1.

□

(b) Show that 0 is a weight of highest weight module $L(\lambda)$ if and only if λ is a sum of roots.

Proof. The weights of $L(\lambda)$ are of the form $\gamma = \lambda - \sum_{\alpha\in R^+}\ell_\alpha\alpha$ with $\ell_\alpha \in \mathbb{Z}_{\geq 0}$. So $\gamma = 0$ exactly when $\lambda = \sum_{\alpha\in R^+}\ell_\alpha\alpha$, i.e. when λ is the sum of roots. □

(2) **Type A_r stuff.** Analyze the standard representation of $\mathfrak{sl}_3$.

(a) What are the primitive elements?

(b) What is/are the weight/weights of the action of $\mathfrak{h}$ on the primitive elements (in terms of ω_1 and ω_2)?

(c) What is the standard representation isomorphic to (in terms of highest weight modules)?

(d) Draw a picture of the weights and verify that the dimension is correct.

(e) What is the standard representation (in terms of highest weight modules) of $\mathfrak{sl}_{r+1}$ in general?

In the standard representation of $\mathfrak{sl}_3$,

$$h_1 = h_{\varepsilon_1-\varepsilon_2} = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} \qquad \text{and} \qquad h_2 = h_{\varepsilon_2-\varepsilon_3} = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},$$

So the standard basis $v_1=(1,0,0), v_2=(0,1,0), v_3=(0,0,1)$, of $V=\mathbb{C}^3$ is a weight basis. Further,

$$x_{\alpha_1} = E_{1,2} = y_{-\alpha_1}^T, x_{\alpha_2} = E_{2,3} = y_{-\alpha_2}^T, \qquad \text{and} \qquad x_{\alpha_1+\alpha_2} = E_{1,3} = y_{-\alpha_1-\alpha_2}^T,$$

and so the primitive elements are those simultaneously annihilated by $E_{1,2}, E_{2,3}$, and $E_{1,2}$. Thus v_1 is the unique (up to scaling) primitive element of V.

To calculate the weight λ of the action of $\mathfrak{h}$ on v^+, note that the action of $\mathfrak{h}$ on $v^+=v_1$ is generated by $h_1v^+=v^+$ (so that $\lambda(h_1)=1$) and $h_2v^+=0$ (so $\lambda(h_2)=0$.) The fundamental weights of $\mathfrak{sl}_3$ are given by $\omega_1 = \varepsilon_1 - \frac{1}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$ and $\omega_2 = \varepsilon_1+\varepsilon_2 - \frac{2}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$. Write $\lambda = a_1\omega_1 + a_2\omega_2$ so that

$$\lambda(h_1) = \langle \varepsilon_1 - \varepsilon_2, a_1\omega_1 + a_2\omega_2\rangle = a_1$$

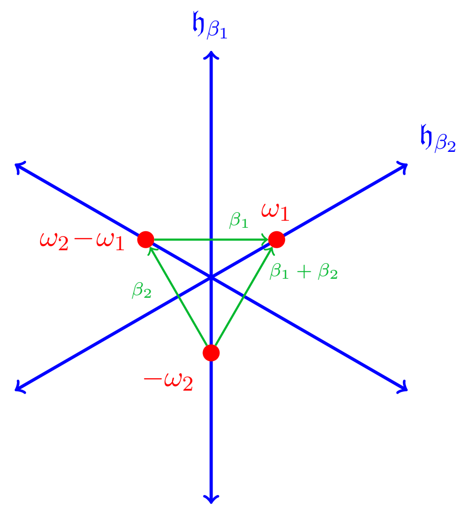
and

$$\lambda(h_2) = \langle \varepsilon_2 - \varepsilon_3, a_1\omega_1 + a_2\omega_2\rangle = a_2$$

so that $a_1 = 1$ and $a_2 = 0$, and $\lambda = \omega_1$.

So $V = L(\omega_1)$.

For the picture, note that ω_1 is on a hyperplane, so that $W\omega_1$ has only three points. Also, these points are single root shifts of each other, so there are no other weights in $L(\omega_1)$. So since $\dim(V_{s_\alpha(\lambda)}) = \dim(V_\lambda) = 1$, we have $\dim(L(\omega_1)) = 3$ as desired:



For general r, the standard basis is still a weight basis, and v_1 is the unique (up to scaling) element which is simultaneously annihilated by $\{E_{i,j} \mid 1 \le i < j \le r+1\}$, so V is simple and $v_1 = v^+$ is the primitive element generating V. The action of $\mathfrak{h}$ on v^+ is given by $h_\ell v^+ = \delta_{1,\ell} v^+$, so again v^+ has weight ω_1. So $V = L(\omega_1)$.

(3) **Type C_r stuff.**

(a) Give a base for the set of roots of type C_r, and calculate the corresponding simple co-roots and fundamental weights.

For type C_r, the roots are given by $R = \{\pm 2\varepsilon_k, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \le k, i < j \le r\}$ with $\mathfrak{h}^* = \mathbb{C}R = \mathbb{C}^r$ with orthonormal basis $\{\varepsilon_1, \dots, \varepsilon_r\}$ with respect to the form induced by trace form on the standard representation. So one base for R is $B = \{\beta_i \mid i = 1, \dots, r\}$, with

$$\beta_r = 2\varepsilon_r \qquad \text{and} \qquad \beta_i = \varepsilon_i - \varepsilon_{i+1} \quad \text{for } i = 1, \dots, r-1.$$

Then $R^+ = \{\varepsilon_k, \varepsilon_i \pm \varepsilon_j \mid 1 \le k, i < j \le r\}$. For $1 \le i \le r-1$, $\langle \beta_i, \beta_i\rangle = 2$, so $\beta_i^\vee = \beta_i$. For $i = r$, $\langle \beta_r, \beta_r\rangle = 4$, so $\beta_r^\vee = \frac{1}{2}\beta_r = \varepsilon_r$. So the fundamental weights are given by

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for } i = 1, \dots, r.$$

(b) Give two examples of highest weight modules for C_2 for which every weight space has multiplicity 1 (and justify how you know every weight space has multiplicity 1).

The trivial representation $L(0)$ is one-dimensional by definition. Just as in part (2), the representations $L(\omega_1)$ is a 4-dimensional module1 whose weights are the W-orbit of the highest weight, and so the weight spaces all have the same multiplicity as the top weight, namely, 1.
Note that $L(\omega_2) = L(\varepsilon_1 + \varepsilon_2)$ has 0 as a non-trivial weight (since $\varepsilon_1 + \varepsilon_2$ is also a root), so $L(\omega_2)$ does not work here.