Exercise 5: Some things about weights and representations.

1. Let $g$ be a finite-dimensional complex semisimple Lie algebra.
   (a) Show that if $L(\lambda)$ and $L(\mu)$ are highest weight modules (of weights $\lambda$ and $\mu$), show that $L(\lambda) \otimes L(\mu)$ has $L(\lambda + \mu)$ as a submodule with multiplicity 1. (Think about primitive elements)

   Proof. In a highest weight module $L(\mu)$, the spanning set of $\{ yv^+_\mu \mid y = y_1^{m_1} \cdots y_l^{m_l}, y_i \in g_{-\alpha_i} \}$ is a weight spanning set, where the weight of $yv^+_\mu$ is $\mu - \sum_i m_i \alpha_i$. So the set $\{ yv^+_\lambda \otimes y'v^+_\mu \mid y = y_1^{m_1} \cdots y_l^{m_l}, y' = y'_1^{n_1} \cdots y'_l^{n_l} \}$ is a weight spanning set for $L(\lambda) \otimes L(\mu)$, where the weight of $v^+_\lambda \otimes v^+_\mu$ is

   $$(\lambda - \sum_i n_i \alpha_i) + (\mu - \sum_i m_i \alpha_i) = \lambda + \mu - \sum_i (n_i + m_i) \alpha_i.$$ 

   So the multiplicity of the weight space or weight $\lambda + \mu$ is one, and is generated by $v^+_\lambda \otimes v^+_\mu$. Further, $v^+_\lambda \otimes v^+_\mu$ is primitive. So $L(\lambda + \mu)$ is a submodule of $L(\lambda) \otimes L(\mu)$ with multiplicity 1. □

   (b) Show that 0 is a weight of highest weight module $L(\lambda)$ if and only if $\lambda$ is a sum of roots.

   Proof. The weights of $L(\lambda)$ are of the form $\gamma = \lambda - \sum_{\alpha \in R^+} \ell_\alpha \alpha$ with $\ell_\alpha \in \mathbb{Z}_{\geq 0}$. So $\gamma = 0$ exactly when $\lambda = \sum_{\alpha \in R^+} \ell_\alpha \alpha$, i.e. when $\lambda$ is the sum of roots. □

2. Type $A_r$ stuff. Analyze the standard representation of $\mathfrak{sl}_3$.
   (a) What are the primitive elements?
   (b) What is/are the weight/weights of the action of $\mathfrak{h}$ on the primitive elements (in terms of $\omega_1$ and $\omega_2$)?
   (c) What is the standard representation isomorphic to (in terms of highest weight modules)?
   (d) Draw a picture of the weights and verify that the dimension is correct.
   (e) What is the standard representation (in terms of highest weight modules) of $\mathfrak{sl}_{r+1}$ in general?

   In the standard representation of $\mathfrak{sl}_3$,

   $$h_1 = h_{\varepsilon_1 - \varepsilon_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad h_2 = h_{\varepsilon_2 - \varepsilon_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

   So the standard basis $v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1)$, of $V = \mathbb{C}^3$ is a weight basis. Further,

   $$x_{\alpha_1} = E_{1,2} = y^T_{-\alpha_1}, x_{\alpha_2} = E_{2,3} = y^T_{-\alpha_2}, \quad \text{and} \quad x_{\alpha_1 + \alpha_2} = E_{1,3} = y^T_{-\alpha_1 - \alpha_2},$$

   and so the primitive elements are those simultaneously annihilated by $E_{1,2}, E_{2,3},$ and $E_{1,3}$. Thus $v_1$ is the unique (up to scaling) primitive element of $V$.

   To calculate the weight $\lambda$ of the action of $\mathfrak{h}$ on $v^+$, note that the action of $\mathfrak{h}$ on $v^+$ is generated by $h_1 v^+ = v^+$ (so that $\lambda(h_1) = 1$) and $h_2 v^+ = 0$ (so $\lambda(h_2) = 0$.) The fundamental
weights of $\mathfrak{sl}_3$ are given by $\omega_1 = \varepsilon_1 - \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ and $\omega_2 = \varepsilon_1 + \varepsilon_2 - \frac{2}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$. Write $\lambda = a_1\omega_1 + a_2\omega_2$ so that

$$\lambda(h_1) = \langle \varepsilon_1 - \varepsilon_2, a_1\omega_1 + a_2\omega_2 \rangle = a_1$$

and

$$\lambda(h_2) = \langle \varepsilon_2 - \varepsilon_3, a_1\omega_1 + a_2\omega_2 \rangle = a_2$$

so that $a_1 = 1$ and $a_2 = 0$, and $\lambda = \omega_1$.

So $V = L(\omega_1)$.

For the picture, note that $\omega_1$ is on a hyperplane, so that $W\omega_1$ has only three points. Also, these points are single root shifts of each other, so there are no other weights in $L(\omega_1)$. So since $\dim(V_{s_\alpha}(\lambda)) = \dim(V_{\lambda}) = 1$, we have $\dim(L(\omega_1)) = 3$ as desired:

For general $r$, the standard basis is still a weight basis, and $v_1$ is the unique (up to scaling) element which is simultaneously annihilated by $\{E_{i,j} \mid 1 \leq i < j \leq r + 1\}$, so $V$ is simple and $v_1 = v^+$ is the primitive element generating $V$. The action of $\mathfrak{h}$ on $v^+$ is given by $h_{\alpha}v^+ = \delta_{1,\alpha}v^+$, so again $v^+$ has weight $\omega_1$. So $V = L(\omega_1)$.

(3) **Type $C_r$ stuff.**

(a) Give a base for the set of roots of type $C_r$, and calculate the corresponding simple co-roots and fundamental weights.

For type $C_r$, the roots are given by $R = \{\pm 2\varepsilon_k, \pm (\varepsilon_i \pm \varepsilon_j) \mid 1 \leq k, i < j \leq r\}$ with $\mathfrak{h}^* = \mathbb{C}R = \mathbb{C}^r$ with orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_r\}$ with respect to the form induced by trace form on the standard representation. So one base for $R$ is $B = \{\beta_i \mid i = 1, \ldots, r\}$, with

$$\beta_r = 2\varepsilon_r$$
and
$$\beta_i = \varepsilon_i - \varepsilon_{i+1} \quad \text{for} \quad i = 1, \ldots, r - 1.$$

Then $R^+ = \{\varepsilon_k, \varepsilon_i \pm \varepsilon_j \mid 1 \leq k, i < j \leq r\}$. For $1 \leq i \leq r - 1$, $\langle \beta_i, \beta_i \rangle = 2$, so $\beta_i^\vee = \beta_i$. For $i = r$, $\langle \beta_r, \beta_r \rangle = 4$, so $\beta_r^\vee = \frac{1}{2}\beta_r = \varepsilon_r$. So the fundamental weights are given by

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for} \quad i = 1, \ldots, r.$$
(b) Give two examples of highest weight modules for $C_2$ for which every weight space has multiplicity 1 (and justify how you know every weight space has multiplicity 1).

The trivial representation $L(0)$ is one-dimensional by definition. Just as in part (2), the representations $L(\omega_1)$ is a 4-dimensional module whose weights are the $W$-orbit of the highest weight, and so the weight spaces all have the same multiplicity as the top weight, namely, 1.

Note that $L(\omega_2) = L(\varepsilon_1 + \varepsilon_2)$ has 0 as a non-trivial weight (since $\varepsilon_1 + \varepsilon_2$ is also a root), so $L(\omega_2)$ does not work here.