Math 128: Combinatorial representation theory of complex Lie algebras and related topics

Recommended reading

For the first while:

1. N. Bourbaki, *Elements of Mathematics: Lie Groups and Algebras.*
2. W. Fulton, J. Harris, *Representation Theory: A first course.*
3. J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory.*
4. J. J. Serre, *Complex Semisimple Lie Algebras.*

Later:

5. H. Barcelo, A. Ram, *Combinatorial Representation Theory.*

...among others

The poster child of CRT: the symmetric group

Combinatorial representation theory is the study of representations of algebraic objects, using combinatorics to keep track of the relevant information.

The poster child of CRT: the symmetric group

Combinatorial representation theory is the study of representations of algebraic objects, using combinatorics to keep track of the relevant information.

What are the algebraicic objects?

Let F be your favorite field of characteristic 0. (Really, fix $F = \mathbb{C}$.) Recall that an *algebra* A over F is a vector space over F with an associative multiplication

$$A \otimes A \to A$$

(tensor product over F just means the multiplication is bilinear).

The poster child of CRT: the symmetric group

Combinatorial representation theory is the study of representations of algebraic objects, using combinatorics to keep track of the relevant information.

What are the algebraicic objects?

Let F be your favorite field of characteristic 0. (Really, fix $F = \mathbb{C}$.) Recall that an *algebra* A over F is a vector space over F with an associative multiplication

$$A \otimes A \to A$$

(tensor product over F just means the multiplication is bilinear).

Favorite examples:

1. Group algebras (today)
2. Enveloping algebras of Lie algebras (next)

Representations

A *homomorphism* is a structure-preserving map.
A *representation* of an F-algebra A is a vector space V over F, together with a homomorphism

$$\rho : A \to \mathrm{End}(V) = \{\ F\text{-linear maps } V \to V\ \}.$$

The map (equipped with the vector space) is the representation; the vector space (equipped with the map) is called an A-*module*.

Representations

A *homomorphism* is a structure-preserving map.
A *representation* of an F-algebra A is a vector space V over F, together with a homomorphism

$$\rho : A \to \mathrm{End}(V) = \{\ F\text{-linear maps } V \to V\ \}.$$

The map (equipped with the vector space) is the representation; the vector space (equipped with the map) is called an A-*module*.

Example

The permutation representation of the symmetric group S_n is $V = \mathbb{C}^k = \mathbb{C}\{v_1, \ldots, v_k\}$ together with

$$\rho : S_k \to \mathrm{GL}_k(\mathbb{C}) \qquad \text{by} \qquad \rho(\sigma)v_i = v_{\sigma(i)}.$$

Representations

A *homomorphism* is a structure-preserving map.
A *representation* of an F-algebra A is a vector space V over F, together with a homomorphism

$$\rho : A \to \mathrm{End}(V) = \{\ F\text{-linear maps } V \to V\ \}.$$

The map (equipped with the vector space) is the representation; the vector space (equipped with the map) is called an A-*module*.

Example

The permutation representation of the symmetric group S_n is $V = \mathbb{C}^k = \mathbb{C}\{v_1, \ldots, v_k\}$ together with

$$\rho : S_k \to \mathrm{GL}_k(\mathbb{C}) \qquad \text{by} \qquad \rho(\sigma)v_i = v_{\sigma(i)}.$$

A *simple* module is a module with no nontrivial invariant subspaces.

Permutation representation of S_3

On the basis $\{v_1, v_2, v_3\}$:

$$1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (132) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad (13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Permutation representation of S_3

On the basis $\{v_1, v_2, v_3\}$:

$$1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (132) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad (13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

On the basis $\{v, w_2, w_3\}$:

$$1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (132) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix} \qquad (13) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

An algebra is *semisimple* if all of its modules decompose into the sum of simple modules. We like these because their isomorphic to a sum of their simple modules.

An algebra is *semisimple* if all of its modules decompose into the sum of simple modules. We like these because their isomorphic to a sum of their simple modules.

Example

The group algebra of a group G over a field F is semisimple iff $\mathrm{char}(F)$ does not divide $|G|$. So group algebras over $\mathbb{C}$ are all semisimple.

An algebra is *semisimple* if all of its modules decompose into the sum of simple modules. We like these because their isomorphic to a sum of their simple modules.

Example

The group algebra of a group G over a field F is semisimple iff $\mathrm{char}(F)$ does not divide $|G|$. So group algebras over $\mathbb{C}$ are all semisimple.

Theorem

For a finite group G, the simple G-modules are in bijection with conjugacy classes of G.

An algebra is *semisimple* if all of its modules decompose into the sum of simple modules. We like these because their isomorphic to a sum of their simple modules.

Example

The group algebra of a group G over a field F is semisimple iff $\mathrm{char}(F)$ does not divide $|G|$. So group algebras over $\mathbb{C}$ are all semisimple.

Theorem

For a finite group G, the simple G-modules are in bijection with conjugacy classes of G.

Proof.

Use (A) class sums, or (B) character theory. $\square$

An algebra is *semisimple* if all of its modules decompose into the sum of simple modules. We like these because their isomorphic to a sum of their simple modules.

Example

The group algebra of a group G over a field F is semisimple iff $\mathrm{char}(F)$ does not divide $|G|$. So group algebras over $\mathbb{C}$ are all semisimple.

Theorem

For a finite group G, the simple G-modules are in bijection with conjugacy classes of G.

Proof.

Use (A) class sums, or (B) character theory. □

Either way, this is not a particularly satisfying bijection, since it doesn't say "given representation X, here's conjugacy class Y, and vice versa."

Character theory

A *character* χ of a group G corresponding to a representation ρ is a homomorphism

$$\chi_\rho : G \to \mathbb{C} \qquad \text{defined by} \qquad \chi_\rho : g \to \mathrm{tr}(\rho(g)).$$

Character theory

A *character* χ of a group G corresponding to a representation ρ is a homomorphism

$$\chi_\rho : G \to \mathbb{C} \qquad \text{defined by} \qquad \chi_\rho : g \to \mathrm{tr}(\rho(g)).$$

Nice facts about characters:

(1) They're *class functions*.

Character theory

A *character* χ of a group G corresponding to a representation ρ is a homomorphism

$$\chi_\rho : G \to \mathbb{C} \qquad \text{defined by} \qquad \chi_\rho : g \to \mathrm{tr}(\rho(g)).$$

Nice facts about characters:

(1) They're *class functions*.

(2) They satisfy nice relations like

$$\chi_{\rho \oplus \psi} = \chi_\rho + \chi_\psi$$
$$\chi_{\rho \otimes \psi} = \chi_\rho \chi_\psi$$

Character theory

A *character* χ of a group G corresponding to a representation ρ is a homomorphism

$$\chi_\rho : G \to \mathbb{C} \qquad \text{defined by} \qquad \chi_\rho : g \to \mathrm{tr}(\rho(g)).$$

Nice facts about characters:

(1) They're *class functions*.

(2) They satisfy nice relations like

$$\chi_{\rho \oplus \psi} = \chi_\rho + \chi_\psi$$
$$\chi_{\rho \otimes \psi} = \chi_\rho \chi_\psi$$

(3) The simple characters form an orthonormal basis for the class functions on G.

Simple symmetric group modules

Conjugacy classes of the symmetric group are given by cycle type.

Example: S_4

$$(a)(b)(c)(d) \qquad (ab)(c)(d) \qquad (ab)(cd) \qquad (abc)(d) \qquad (abcd)$$

Simple symmetric group modules

Conjugacy classes of the symmetric group are given by cycle type.

Example: S_4

$$(a)(b)(c)(d) \qquad (ab)(c)(d) \qquad (ab)(cd) \qquad (abc)(d) \qquad (abcd)$$

Cycle types of permutations of k are in bijection with *partitions* $\lambda \vdash k$:

$$\lambda = (\lambda_1, \lambda_2, \dots) \quad \text{with } \lambda_1 \geq \lambda_2 \geq \dots, \quad \lambda_i \in \mathbb{Z}_{\geq 0}$$
$$\text{and } \lambda_1 + \lambda_2 + \dots = k.$$

Simple symmetric group modules

Conjugacy classes of the symmetric group are given by cycle type.

Example: S_4

$(a)(b)(c)(d)$	$(ab)(c)(d)$	$(ab)(cd)$	$(abc)(d)$	$(abcd)$
$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)

Cycle types of permutations of k are in bijection with *partitions* $\lambda \vdash k$:

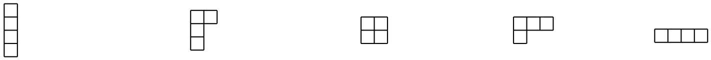
$$\lambda = (\lambda_1, \lambda_2, \dots) \quad \text{with } \lambda_1 \geq \lambda_2 \geq \dots, \quad \lambda_i \in \mathbb{Z}_{\geq 0}$$
$$\text{and } \lambda_1 + \lambda_2 + \cdots = k.$$

Simple symmetric group modules

Conjugacy classes of the symmetric group are given by cycle type.

Example: S_4

$(a)(b)(c)(d)$	$(ab)(c)(d)$	$(ab)(cd)$	$(abc)(d)$	$(abcd)$
$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)



Cycle types of permutations of k are in bijection with *partitions* $\lambda \vdash k$:

$$\lambda = (\lambda_1, \lambda_2, \dots) \quad \text{with } \lambda_1 \geq \lambda_2 \geq \dots, \quad \lambda_i \in \mathbb{Z}_{\geq 0}$$
$$\text{and } \lambda_1 + \lambda_2 + \cdots = k.$$

(Pictures are up-left justified arrangements of boxes with λ_i boxes in the ith row.)

More representation theory of S_k

Some combinatorial facts: (without proof)

More representation theory of S_k

Some combinatorial facts: (without proof)

(1) If S^λ is the module indexed by λ, then

$$\mathrm{Ind}_{S_k}^{S_{k+1}}(S^\lambda) = \bigoplus_{\substack{\mu \vdash k+1 \\ \mu \in \lambda^+}} S^\mu \qquad \text{and } \mathrm{Res}_{S_{k-1}}^{S_k}(S^\lambda) = \bigoplus_{\substack{\mu \vdash k-1 \\ \lambda \in \mu^+}} S^\mu$$

where λ^+ is the set of partitions that look like λ plus a box.

More representation theory of S_k

Some combinatorial facts: (without proof)

(1) If S^λ is the module indexed by λ, then

$$\mathrm{Ind}_{S_k}^{S_{k+1}}(S^\lambda) = \bigoplus_{\substack{\mu \vdash k+1 \\ \mu \in \lambda^+}} S^\mu \qquad \text{and } \mathrm{Res}_{S_{k-1}}^{S_k}(S^\lambda) = \bigoplus_{\substack{\mu \vdash k-1 \\ \lambda \in \mu^+}} S^\mu$$

where λ^+ is the set of partitions that look like λ plus a box.

(2) The basis for S^λ is indexed by downward-moving paths from $\emptyset$ to λ.

More representation theory of S_k

Some combinatorial facts: (without proof)

(1) If S^λ is the module indexed by λ, then

$$\mathrm{Ind}_{S_k}^{S_{k+1}}(S^\lambda) = \bigoplus_{\substack{\mu \vdash k+1 \\ \mu \in \lambda^+}} S^\mu \qquad \text{and } \mathrm{Res}_{S_{k-1}}^{S_k}(S^\lambda) = \bigoplus_{\substack{\mu \vdash k-1 \\ \lambda \in \mu^+}} S^\mu$$

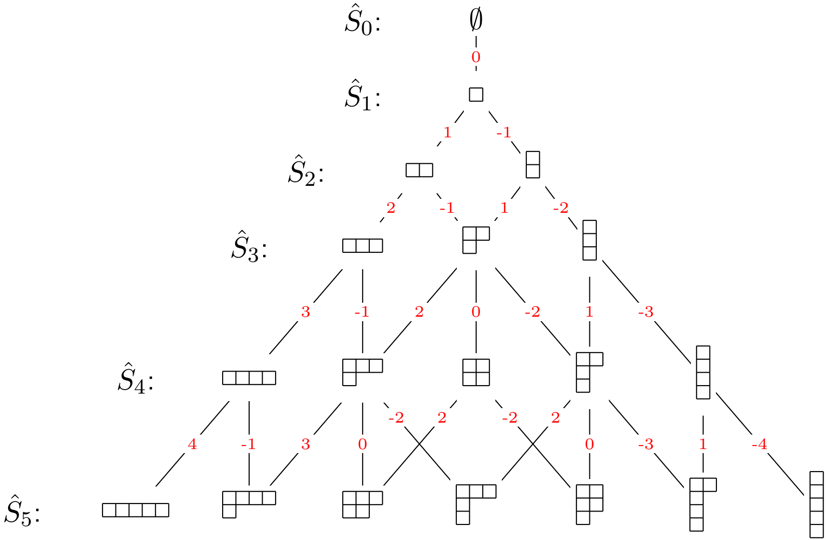
where λ^+ is the set of partitions that look like λ plus a box.

(2) The basis for S^λ is indexed by downward-moving paths from $\emptyset$ to λ.

(3) The matrix entries for ρ_λ are functions of *contents* of added boxes: the *content* of a box b in row i and column j of a partition as

$$c(b) = j - i, \qquad \text{the diagonal number of } b.$$

(The matrix entries for the transposition $(i\ i+1)$ are functions of the values on the edges between levels $i-1$, i, and $i+1$.)



$\hat{S}_0$:
$\emptyset$
0
$\hat{S}_1$:
1
-1
$\hat{S}_2$:
2
-1
1
-2
$\hat{S}_3$:
3
-1
2
0
-2
1
-3
$\hat{S}_4$:
4
-1
3
0
-2
2
-2
2
0
-3
1
-4
$\hat{S}_5$: