Math 128: Combinatorial representation theory of complex Lie algebras and related topics
Recommended reading

For the first while:

2. W. Fulton, J. Harris, *Representation Theory: A first course*.
3. J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*.

Later:

5. H. Barcelo, A. Ram, *Combinatorial Representation Theory*.

...among others
The poster child of CRT: the symmetric group

Combinatorial representation theory is the study of representations of algebraic objects, using combinatorics to keep track of the relevant information.
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What are the algebraic objects?

Let $F$ be your favorite field of characteristic 0. (Really, fix $F = \mathbb{C}$.)

Recall that an *algebra* $A$ over $F$ is a vector space over $F$ with an associative multiplication

$$A \otimes A \to A$$

(tensor product over $F$ just means the multiplication is bilinear).
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Favorite examples:

1. Group algebras (today)
2. Enveloping algebras of Lie algebras (next)
Representations

A *homomorphism* is a structure-preserving map. A *representation* of an $F$-algebra $A$ is a vector space $V$ over $F$, together with a homomorphism

$$\rho : A \to \text{End}(V) = \{ \text{ } F\text{-linear maps } V \to V \}.$$  

The map (equipped with the vector space) is the representation; the vector space (equipped with the map) is called an $A$-*module*.
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**Example**

The permutation representation of the symmetric group $S_n$ is $V = \mathbb{C}^k = \mathbb{C}\{v_1, \ldots, v_k\}$ together with

$$\rho : S_k \to \text{GL}_k(\mathbb{C}) \quad \text{by} \quad \rho(\sigma)v_i = v_{\sigma(i)}.$$
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A simple module is a module with no nontrivial invariant subspaces.
Permutation representation of $S_3$

On the basis $\{v_1, v_2, v_3\}$:

$$
1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
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$$(123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (132) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad (13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
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An algebra is *semisimple* if all of its modules decompose into the sum of simple modules. We like these because their isomorphic to a sum of their simple modules.

Example

The group algebra of a group $G$ over a field $F$ is semisimple iff $\text{char}(F)$ does not divide $|G|$. So group algebras over $\mathbb{C}$ are all semisimple.

**Theorem**

For a finite group $G$, the simple $G$-modules are in bijection with conjugacy classes of $G$.

**Proof.**

Use (A) class sums, or (B) character theory. Either way, this is not a particularly satisfying bijection, since it doesn't say "given representation $X$, here's conjugacy class $Y$, and vice versa."
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Character theory

A character $\chi$ of a group $G$ corresponding to a representation $\rho$ is a homomorphism

$$\chi_\rho : G \to \mathbb{C} \quad \text{defined by} \quad \chi_\rho : g \to \text{tr}(\rho(g)).$$
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Nice facts about characters:

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   $$\chi_\rho \oplus \psi = \chi_\rho + \chi_\psi$$
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3. The simple characters form an orthonormal basis for the class functions on $G$. 
Simple symmetric group modules

Conjugacy classes of the symmetric group are given by cycle type.

Example: $S_4$

$$(a)(b)(c)(d) \quad (ab)(c)(d) \quad (ab)(cd) \quad (abc)(d) \quad (abcd)$$
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Cycle types of permutations of $k$: are in bijection with partitions $\lambda \vdash k$:

$$\lambda = (\lambda_1, \lambda_2, \ldots) \quad \text{with} \quad \lambda_1 \geq \lambda_2 \geq \ldots, \quad \lambda_i \in \mathbb{Z}_{\geq 0}$$

and $\lambda_1 + \lambda_2 + \cdots = k$. 
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(Pictures are up-left justified arrangements of boxes with $\lambda_i$ boxes in the $i$th row.)
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Some combinatorial facts: (without proof)
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(1) If $S^\lambda$ is the module indexed by $\lambda$, then

$$\text{Ind}_{S_k}^{S_{k+1}}(S^\lambda) = \bigoplus_{\mu \vdash k+1, \mu \in \lambda^+} S^\mu$$

and

$$\text{Res}_{S_{k-1}}^{S_k}(S^\lambda) = \bigoplus_{\mu \vdash k-1, \lambda \in \mu^+} S^\mu$$

where $\lambda^+$ is the set of partitions that look like $\lambda$ plus a box.
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2. The basis for \( S^\lambda \) is indexed by downward-moving paths from \( \emptyset \) to \( \lambda \).
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   where $\lambda^+$ is the set of partitions that look like $\lambda$ plus a box.

2. The basis for $S^\lambda$ is indexed by downward-moving paths from $\emptyset$ to $\lambda$.

3. The matrix entries for $\rho_\lambda$ are functions of *contents* of added boxes: the *content* of a box $b$ in row $i$ and column $j$ of a partition as

   $c(b) = j - i$, \hspace{1cm} the diagonal number of $b$.

   (The matrix entries for the transposition $(i \ i+1)$ are functions of the values on the edges between levels $i - 1$, $i$, and $i + 1$.)