Math 128: Lecture 10

April 16, 2014

Last time:

For any basis B of $\mathfrak{h}^*$ consisting of roots, the spaces

$$\mathfrak{h}^*_{\mathbb{Q}} = \mathbb{Q}B \qquad \text{and} \qquad \mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{h}^*_{\mathbb{Q}}$$

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$$\mathfrak{h}_\alpha = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \lambda, \alpha \rangle = 0\}.$$

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$$\sigma_\alpha : \lambda \mapsto \lambda - 2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle}\alpha,$$

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The group W generated by $\{\sigma_\alpha \mid \alpha \in R^+\}$ is called the *Weyl group* associated to $\mathfrak{g}$.

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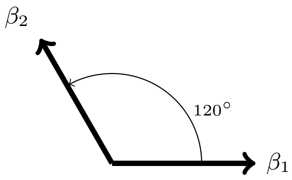

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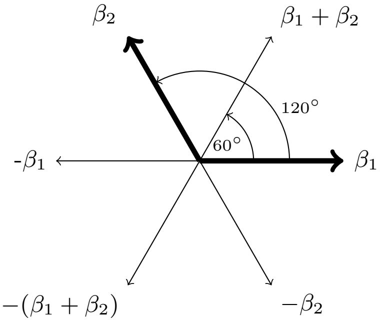

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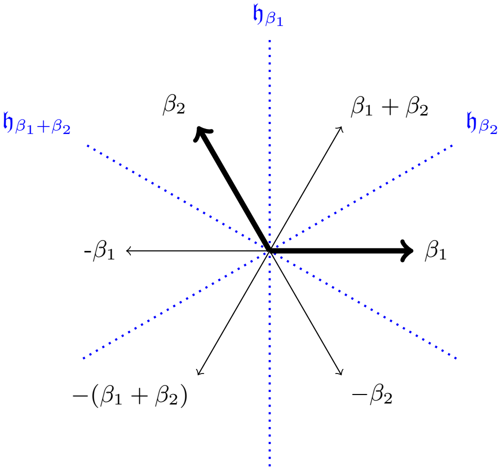

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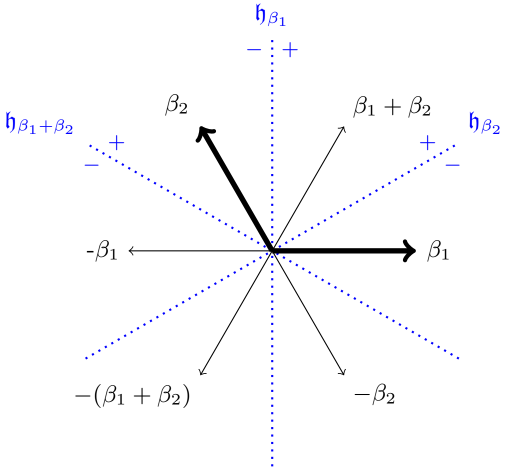

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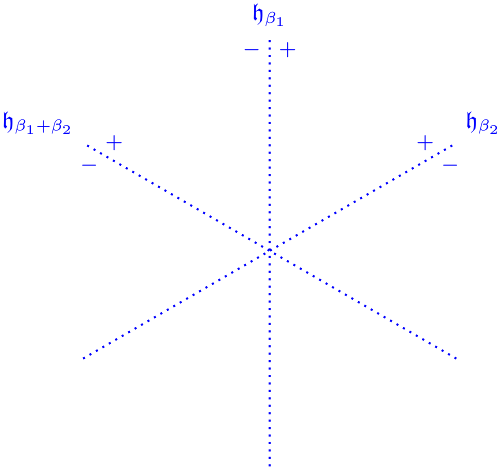

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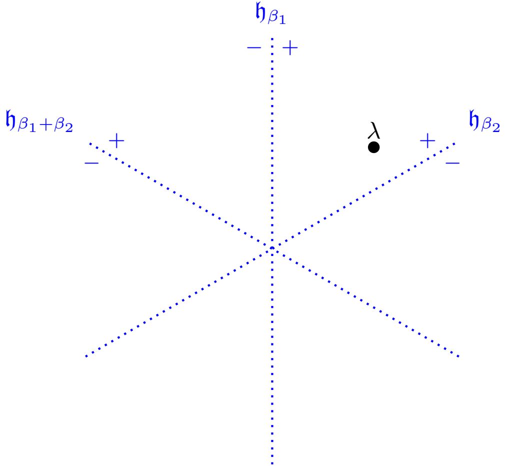

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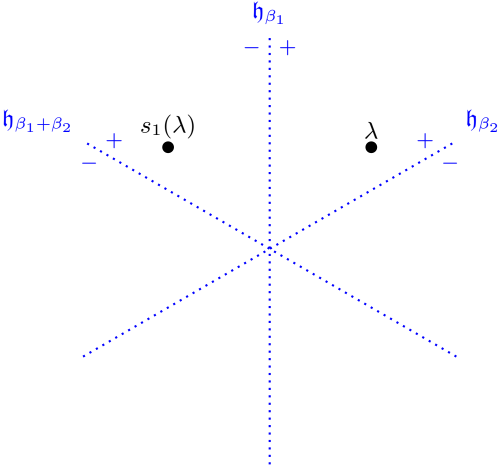

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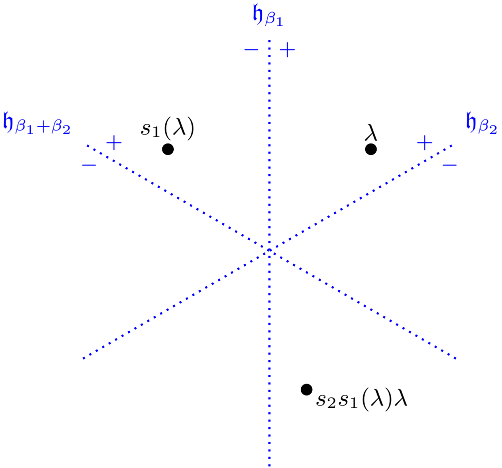

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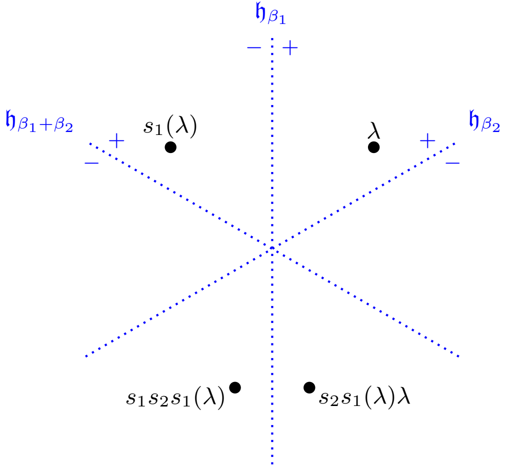

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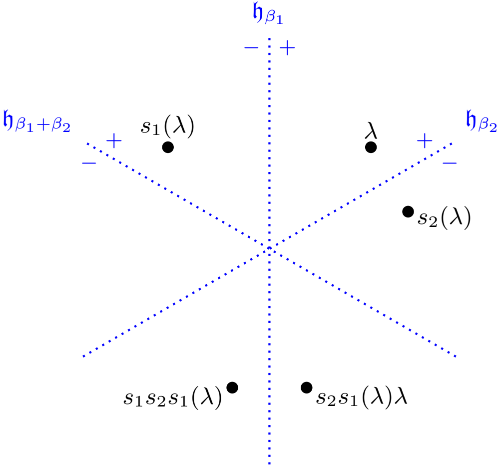

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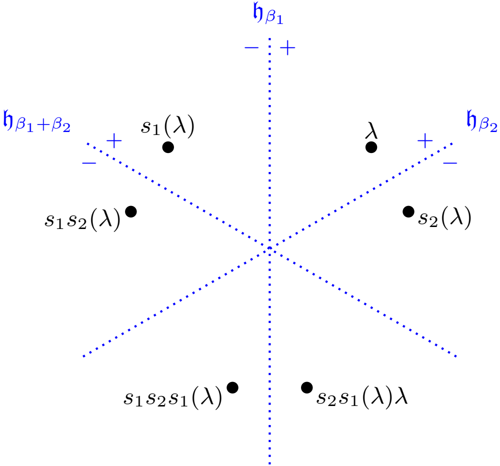

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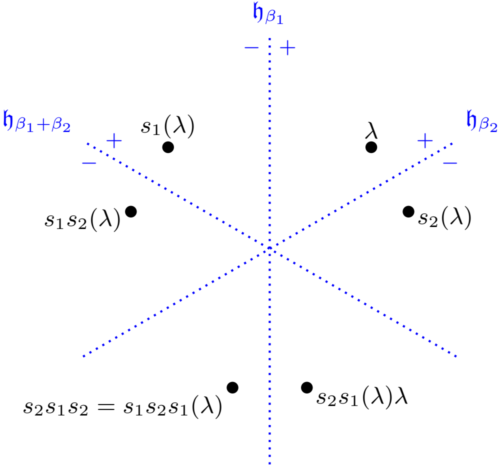

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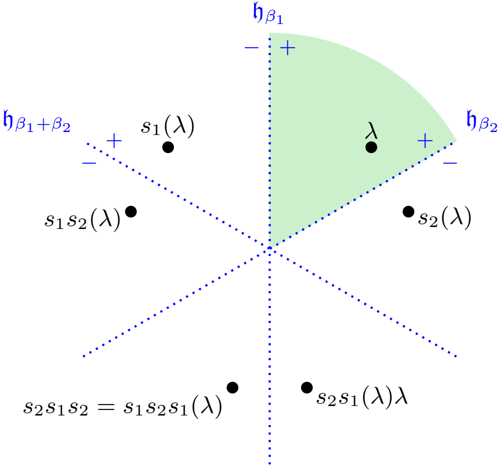

The *positive* side of a hyperplane $\mathfrak{h}_\alpha$ is the side corresponding to whichever of $\pm\alpha$ is in R^+. The *fundamental chamber* is the region of $\mathfrak{h}_{\mathbb{R}}^*$ that is on the positive side of every $\mathfrak{h}_\alpha$, $\alpha \in R$. Every element of $\mathfrak{h}_{\mathbb{R}}^*$ is in the W-orbit of the closure of the fundamental chamber.

Recall classifying finite-dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-modules V:

(1) h has at least one weight vector $v \in V$. Use $hx = xh + [h, x]$ to show that $\{x^\ell v^+ \mid \ell \in \mathbb{Z}_{\geq 0}\}$ are also w.v.'s with distinct weights.

(2) Since the weights of h on the $x^\ell v$'s are distinct, the non-zero $x^\ell v$'s are distinct. So since V is f.d., there must be $0 \neq v^+ \in V$ with

$$xv^+ = 0 \qquad \text{and} \qquad hv^+ = \mu v^+ \text{ for some } \mu \in \mathbb{C}.$$

The vector v^+ is called primitive or a highest weight vector.

(3) Use $hy = yh + [h, y]$ to show that $\{y^\ell v^+ \mid \ell \in \mathbb{Z}_{\geq 0}\}$ are also weight vectors with distinct weights. So again, since V is finite-dimensional, there must be some $d \in \mathbb{Z}_{\geq 0}$ with $y^d v^+ \neq 0$ and $y^{d+1} v^+ = 0$.

(4) Use $xy = yx + h$ to show $xy^\ell v^+ = \ell(\mu - (\ell - 1))$, so that $V = \{y^\ell v^+ \mid \ell = 0, 1, \ldots, d\}$.

(5) Looking at the $(d+1, d+1)$ entry of h, use $[x, y] = h$ to show $\mu = d$.

Finite dimensional representations of $\mathfrak{g}$

New strategy:

Replace x with $\mathfrak{n}^+$, y with $\mathfrak{n}^-$, and h with $\mathfrak{h}$.

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$$hv^+ = \lambda(h)v^+ \qquad \text{and} \qquad xv^+ = 0$$

for some $\lambda \in \mathfrak{h}^*$ and all $h \in \mathfrak{h}$, $x \in \mathfrak{n}^+$.

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A *base* B for a set of roots R is a subset of R forming a basis of $\mathfrak{h}^*$ which additionally satisfies

$$\alpha = \pm \sum_{\beta \in B} z_\beta \beta \text{ with } z_\beta \in \mathbb{Z}_{\geq 0} \qquad \text{for all } \alpha \in R.$$

Given a base B, let $R^+ = R \cap \mathbb{Z}_{\geq 0} B$. (We will prove the existence of a base for R later, but we take existence for granted for now.)

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Goal 1: Establish $xv = 0$ for all but finitely many words $x = x_1 \cdots x_m$ with $\alpha_i \in R^+$.