Math 128: Lecture 10

April 16, 2014
Last time:

For any basis $B$ of $\mathfrak{h}^*$ consisting of roots, the spaces

$$\mathfrak{h}_Q^* = \mathbb{Q}B \quad \text{and} \quad \mathfrak{h}_R^* = \mathbb{R} \otimes \mathbb{Q} \mathfrak{h}_Q^*$$

are Euclidean with inner product given by the Killing form (or any positive rational/real scaling thereof).
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Let $\mathfrak{h}_\alpha$ be the hyperplane in $\mathfrak{h}^*_\mathbb{R}$ given by

$$\mathfrak{h}_\alpha = \{ \lambda \in \mathfrak{h}^*_\mathbb{R} \mid \langle \lambda, \alpha \rangle = 0 \}.$$ 

Let $\sigma_\alpha : \mathfrak{h}^*_\mathbb{R} \rightarrow \mathfrak{h}^*_\mathbb{R}$, given by

$$\sigma_\alpha : \lambda \mapsto \lambda - 2 \frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \alpha,$$

be the reflection of weights across the hyperplane $\mathfrak{h}_\alpha$. This map sends roots to roots.
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be the reflection of weights across the hyperplane $\mathfrak{h}_\alpha$. This map sends roots to roots.

The group $W$ generated by $\{ \sigma_\alpha \mid \alpha \in R^+ \}$ is called the Weyl group associated to $\mathfrak{g}$. 

Example: $g = \mathfrak{sl}_3$
Let $B = \{\beta_1, \beta_2\}$ and $R^+ = \{\beta_1, \beta_2, \beta_1 + \beta_2\}$ with $\beta_i = \varepsilon_i - \varepsilon_{i+1}$
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The *positive* side of a hyperplane $h_\alpha$ is the side corresponding to whichever of $\pm \alpha$ is in $R^+$. 

![Diagram of hyperplanes and fundamental chamber](image-url)
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$h_{\beta_1}$

$h_{\beta_1 + \beta_2} - + s_1(\lambda)$

$\lambda - + +$

$h_{\beta_2}$
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The positive side of a hyperplane \( h_\alpha \) is the side corresponding to whichever of \( \pm \alpha \) is in \( R^+ \). The fundamental chamber is the region of \( h_\alpha^* \) that is on the positive side of every \( h_\alpha, \alpha \in R \). Every element of \( h_\alpha^* \) is in the \( W \)-orbit of the closure of the fundamental chamber.
Recall classifying finite-dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-modules $V$:

1. $h$ has at least one weight vector $v \in V$. Use $hx = xh + [h, x]$ to show that $\{x^\ell v^+ \mid \ell \in \mathbb{Z}_{\geq 0}\}$ are also w.v.’s with distinct weights.

2. Since the weights of $h$ on the $x^\ell v$’s are distinct, the non-zero $x^\ell v$’s are distinct. So since $V$ is f.d., there must be $0 \neq v^+ \in V$ with

$$xv^+ = 0 \quad \text{and} \quad hv^+ = \mu v^+ \text{ for some } \mu \in \mathbb{C}.$$ 

The vector $v^+$ is called primitive or a highest weight vector.

3. Use $hy = yh + [h, y]$ to show that $\{y^\ell v^+ \mid \ell \in \mathbb{Z}_{\geq 0}\}$ are also weight vectors with distinct weights. So again, since $V$ is finite-dimensional, there must be some $d \in \mathbb{Z}_{\geq 0}$ with $y^dv^+ \neq 0$ and $y^{d+1}v^+ = 0$.

4. Use $xy = yx + h$ to show $xy^\ell v^+ = \ell(\mu - (\ell - 1))$, so that $V = \{y^\ell v^+ \mid \ell = 0, 1, \ldots, d\}$.

5. Looking at the $(d+1, d+1)$ entry of $h$, use $[x, y] = h$ to show $\mu = d$. 

Finite dimensional representations of $\mathfrak{g}$

**New strategy:**
Replace $x$ with $\mathfrak{n}^+$, $y$ with $\mathfrak{n}^-$, and $h$ with $\mathfrak{h}$.
Let $V$ be a finite-dimensional $\mathfrak{g}$-module.
Finite dimensional representations of $g$

New strategy:
Replace $x$ with $n^+$, $y$ with $n^-$, and $h$ with $\mathfrak{h}$.
Let $V$ be a finite-dimensional $g$-module.
Look for a highest weight vector (a primitive element), i.e. $v^+$ satisfying

$$hv^+ = \lambda(h)v^+ \quad \text{and} \quad xv^+ = 0$$

for some $\lambda \in \mathfrak{h}^*$ and all $h \in \mathfrak{h}$, $x \in n^+$. 

Show $V = U_{n^-v^+}$.

Classify $\lambda$ and the resulting structure.

A base $B$ for a set of roots $R$ is a subset of $R$ forming a basis of $\mathfrak{h}^*$ which additionally satisfies

$$\alpha = \pm \sum_{\beta \in B} z_{\beta} \beta$$

with $z_{\beta} \in \mathbb{Z} \geq 0$ for all $\alpha \in R$.

Given a base $B$, let $R^+ = R \cap \mathbb{Z} \geq 0$.

(We will prove the existence of a base for $R$ later, but we take existence for granted for now.)
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Show $V = U^n^+ v^+$. 

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Show $V = Un^-v^+$.
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$$\alpha = \pm \sum_{\beta \in B} z_\beta \beta \quad \text{with} \quad z_\beta \in \mathbb{Z}_{\geq 0} \quad \text{for all} \quad \alpha \in R.$$  

Given a base $B$, let $R^+ = R \cap \mathbb{Z}_{\geq 0}B$. (We will prove the existence of a base for $R$ later, but we take existence for granted for now.)
Finite dimensional representations of $\mathfrak{g}$

Let $V$ be a finite-dimensional $\mathfrak{g}$-module. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, so that the elements of $\mathfrak{h}$ are simultaneously diagonalizable.
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Let \( \mathfrak{h} \) be a Cartan subalgebra of \( g \), so that the elements of \( \mathfrak{h} \) are simultaneously diagonalizable. So as a \( \mathfrak{h} \)-module,

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V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_{\lambda} \quad \text{where} \quad V_{\lambda} = \{ v \in V \mid hv = \lambda(h)v \}.
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For $v \in V_\lambda$, $h \in \mathfrak{h}$, and $x \in \mathfrak{g}_\alpha$ for some $\alpha \in R$,

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So for $\alpha_i \in R$ and $x_i \in \mathfrak{g}_{\alpha_i}$,

$$hx_1 \cdots x_m v = \left( \lambda(h) + \sum_{i=1}^{m} \alpha_i(h) \right) x_1 \cdots x_m v. \quad (\ast)$$
Finite dimensional representations of \( \mathfrak{g} \)

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So for \( \alpha_i \in R \) and \( x_i \in \mathfrak{g}_{\alpha_i}, \)

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hx_1 \cdots x_m v = \left( \lambda(h) + \sum_{i=1}^{m} \alpha_i(h) \right) x_1 \cdots x_m v. \quad (\ast)
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**Goal 1:** Establish \( xv = 0 \) for all but finitely many words \( x = x_1 \cdots x_m \) with \( \alpha_i \in R^+ \).