Let $V$ be a finite-dimensional simple $g$-module. Taking $\mathfrak{sl}_2$ as a model, we will classify $V$ as follows:

**Step 1:** Show that for any weight vector $v$, $xv$ is also a weight vector for $x$ a monomial in $U\mathfrak{n}^+$. 

**Step 2:** Show the weights of $xv$ are distinct (enough) so that there exists a $v^+ \in V$ with

\[ n^+ v^+ = 0 \quad \text{and} \quad h v^+ = \mu(h)v^+ \text{ for some } \mu \in \mathfrak{h}^*. \]

**Step 3:** Show $yv^+$ is a weight vector for all monomials $y \in U\mathfrak{n}^-$. 

**Step 4:** Show $xyv^+ \in U\mathfrak{n}^-v^+$ so that $V = U\mathfrak{h}^-v^+$. 

**Step 5:** Find a good basis for $V$. 

**Step 6:** Classify $V$ in terms of $\mu$. 

Last time:

Fix a base $B = \{\beta_1, \ldots, \beta_r\}$ and $R^+ = R \cap \mathbb{Z}_{\geq 0}B$.
Let $V$ be a finite-dimensional simple $g$-module.
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Let $V$ be a finite-dimensional simple $g$-module.

Step 1: For $v \in V_\lambda$, $\alpha_i \in R$, $x_i \in g_{\alpha_i}$, and $h \in \mathfrak{h}$,

$$hx_1 \cdots x_m v = \left(\lambda(h) + \sum_{i=1}^{m} \alpha_i(h)\right) x_1 \cdots x_m v. \quad (*)$$
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Define $\Omega = \{\omega_1, \ldots, \omega_r\}$ by $\langle \beta_i, \omega_j \rangle = c_j \delta_{i,j}$ for some fixed $c_j \in \mathbb{R}_{>0}$. So for every $\alpha \in R^+$,

$$\langle \alpha, \omega_j \rangle = \sum_{i=1}^r z_i \langle \beta_i, \omega_j \rangle = z_j c_j \geq 0,$$

and there is some $\omega \in \Omega$ with $\langle \alpha, \omega \rangle > 0$. 

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So on the basis $\{h_\omega \mid \omega \in \Omega\}$, it is clear that $\lambda + \sum_{i=1}^{m} \alpha_i$ are distinct for distinct collections $x_1, \ldots, x_m$. 

Lemma (Step 2)

There is a highest weight vector $v^+ \in V_\lambda$ satisfying $n^+ v^+ = 0$ and $hv^+ = \mu(h)v^+$ for some $\mu \in h^\ast$. 


Last time:

Fix a base \( B = \{\beta_1, \ldots, \beta_r\} \) and \( R^+ = R \cap \mathbb{Z}_{\geq 0}B \).
Let \( V \) be a finite-dimensional simple \( \mathfrak{g} \)-module.

**Step 1:** For \( v \in V_\lambda, \alpha_i \in R, x_i \in \mathfrak{g}_{\alpha_i} \), and \( h \in \mathfrak{h} \),

\[
hx_1 \cdots x_m v = \left( \lambda(h) + \sum_{i=1}^{m} \alpha_i(h) \right) x_1 \cdots x_m v. \tag{\ast}
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There is a highest weight vector \( v^+ \in V \) satisfying

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\]
For \( v \in V_\lambda, \alpha_i \in R, x_i \in g_{\alpha_i}, \) and \( h \in \mathfrak{h}, \)

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**Step 3:** Show $yv^+$ is a weight vector for all monomials $y$ in $U\mathfrak{n}^-$. 
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**Step 3:** Show $yv^+$ is a weight vector for all monomials $y$ in $Un^-$.  
**Step 4:** Show $xyv^+ \in Un^-v^+$ for all $x \in n^+$ and mon’ls $y \in Un^-$.  

Recall the Birkoff-Witt theorem: Let $B = \{\beta_1, \ldots, \beta_r\}$ be a base of $R$ with $R^+ = \{\alpha_1, \ldots, \alpha_\ell\}$. Then there are bases $\{y_{m_1} \cdots y_{m_\ell} | y_i \in g_{-\alpha_i}, m_i \in \mathbb{Z}_{\geq 0}\}$ of $U^-\mathfrak{g}$, $\{h_{m_1} \beta_1 \cdots h_{m_r} \beta_r | m_i \in \mathbb{Z}_{\geq 0}\}$ of $U^0\mathfrak{g}$, and $\{x_{m_1} \cdots x_{m_\ell} | x_i \in g_{\alpha_i}, m_i \in \mathbb{Z}_{\geq 0}\}$ of $U^+\mathfrak{g}$.  


For \( v \in V_\lambda, \alpha_i \in R, x_i \in g_{\alpha_i}, \) and \( h \in \mathfrak{h}, \)
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**Lemma (Step 2)**

*There is a highest weight vector \( v^+ \in V \) satisfying*

\[
n^+ v^+ = 0 \quad \text{and} \quad hv^+ = \mu(h)v^+ \quad \text{for some} \quad \mu \in \mathfrak{h}^*.\]

**Step 3:** Show \( yv^+ \) is a weight vector for all monomials \( y \) in \( U_{n^-}. \)

**Step 4:** Show \( xyv^+ \in U_{n^-}v^+ \) for all \( x \in n^+ \) and monomials \( y \in U_{n^-}. \)

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\[
\begin{align*}
\{ y_1^{m_1} \cdots y_\ell^{m_\ell} \mid y_i \in g_{-\alpha_i}, m_i \in \mathbb{Z}_{\geq 0} \} & \quad \text{of } U^-, \\
\{ h_1^{m_1} \cdots h_r^{m_r} \mid m_i \in \mathbb{Z}_{\geq 0} \} & \quad \text{of } U^0, \text{ and} \\
\{ x_1^{m_1} \cdots x_\ell^{m_\ell} \mid x_i \in g_{\alpha_i}, m_i \in \mathbb{Z}_{\geq 0} \} & \quad \text{of } U^+. 
\end{align*}
\]
Lemma
Let $V$ be a simple finite-dimensional $\mathfrak{g}$-module.

(a) There is a highest weight vector $v^+ \in V$ satisfying

$$h v^+ = \mu(h) v^+ \text{ for some } \mu \in \mathfrak{h}^*,$$
$$n^+ v^+ = 0, \quad \text{and} \quad U n^- v^+ = V.$$

(b) $V$ is spanned by weight vectors

$$\{y_1^{m_1} \cdots y_\ell^{m_\ell} v^+ \mid m_i \in \mathbb{Z}_{\geq 0}\} \quad \text{with} \quad \mathbb{R}^+ = \{\alpha_1, \ldots, \alpha_\ell\}, \quad \text{and} \quad y_i \in \mathfrak{g}_{-\alpha_i},$$

and $h y v^+ = (\mu - \sum_i m_i \alpha_i)(h) y v^+$ for $y = y_1^{m_1} \cdots y_\ell^{m_\ell}$.

(c) The weight spaces of $V$ are

$$V_\lambda \quad \text{with} \quad \lambda = \mu - \sum_{i=1}^r \ell_i \beta_i, \quad \ell_i \in \mathbb{Z}_{\geq 0},$$

where $B = \{\beta_1, \ldots, \beta_r\}$ is a base for the roots of $\mathfrak{g}$. In particular, $\dim(V_\mu) = 1.$
Structure of highest weight representations

When are highest weight modules simple? When are they isomorphic?

We say an element $v_\mu$ of a $\mathfrak{g}$-module $M$ is a \emph{primitive} element or \emph{highest weight vector} (of weight $\mu \in \mathfrak{h}^*$) if

$$hv_\mu = \mu(h)v_\mu \quad \text{and} \quad n^+v_\mu = 0.$$ 

We call any module generated by a primitive $v_\mu$ a \emph{highest weight module} (of weight $\mu$).
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**Lemma**

Let $M$ be generated by primitive $v_\mu$.

($M$ is not a priori simple or finite-dimensional)

(1) Parts (a)–(c) from the previous lemma hold.

(2) $M$ is indecomposable, and therefore simple.

(3) There is a unique (up to scaling) primitive element of $V$.

(4) Two modules $M^{(\mu)}$ and $M^{(\lambda)}$ generated by primitive elements $v_\mu$ and $v_\lambda$, respectively, are isomorphic if and only if $\mu = \lambda$. 