Math 128: Lecture 12

April 18, 2014

Last time:

So far we have

1. Finite-dimensional simple $\mathfrak{g}$-modules V are highest weight modules, i.e. there is some $v^+ \in V$ satisfying

   $$hv^+ = \mu(h)v^+ \text{ for some } \mu \in \mathfrak{h}^*, \text{ and } h \in \mathfrak{h}, \text{ and } \quad \mathfrak{n}^- v^+ = 0.$$

2. Highest weight modules (of weight μ)
   (a) are simple,
   (b) are pairwise isomorphic if and only if they have the same weight, and
   (c) have top dimension ($\dim(V_\mu)$) equal to 1.

We also know that if $V_\lambda \neq 0$, then

$$\lambda = \mu - \sum_i \ell_i \beta_i \qquad \ell_i \in \mathbb{Z}_{\geq 0}, \beta_i \in B,$$

with for $\alpha_i \in R$, $x_i \in \mathfrak{g}_{\alpha_i}$, and $h \in \mathfrak{h}$,

$$h(x_1 \cdots x_m v^+) = \left(\mu + \sum_{i=1}^m \alpha_i\right)(h)(x_1 \cdots x_m v^+). \qquad (*)$$

Let V be a finite-dimensional highest weight module. For a geometric intuition, we restrict to weights in $\mathfrak{h}_{\mathbb{R}}^*$, though this classification can be extended to $\mathfrak{h}^*$.

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Recall, to each $\alpha \in R^+$, there is a $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$ given by

$$\mathfrak{s}_\alpha = \left\langle x_\alpha, y_\alpha, h_{\alpha^\vee} \mid x_\alpha \in \mathfrak{g}_\alpha, y_\alpha \in \mathfrak{g}_{-\alpha}, h_{\alpha^\vee} = \frac{2}{\langle \alpha, \alpha \rangle} h_\alpha \right\rangle \subseteq \mathfrak{g},$$

and $\langle h_{\alpha^\vee}, h_\alpha \rangle = \alpha(h_{\alpha^\vee}) = \langle \alpha^\vee, \alpha \rangle = 2$.

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If $v_\lambda \in V_\lambda \neq 0$, then $\mathfrak{s}_\alpha v_\lambda$ is a finite-dimensional $\mathfrak{s}_\alpha$-module, with weight spaces $V_{\lambda + \ell\alpha}$ having $h_{\alpha^\vee}$-weight $\langle \lambda + \ell\alpha, \alpha^\vee \rangle = \langle \lambda, \alpha^\vee \rangle + 2\ell$.

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Therefore

$$\{\langle \lambda + \ell\alpha, \alpha^\vee \rangle \mid \ell \in \mathbb{Z}, V_{\lambda+\ell\alpha} \neq 0\} \qquad \text{(counting multiplicities)}$$

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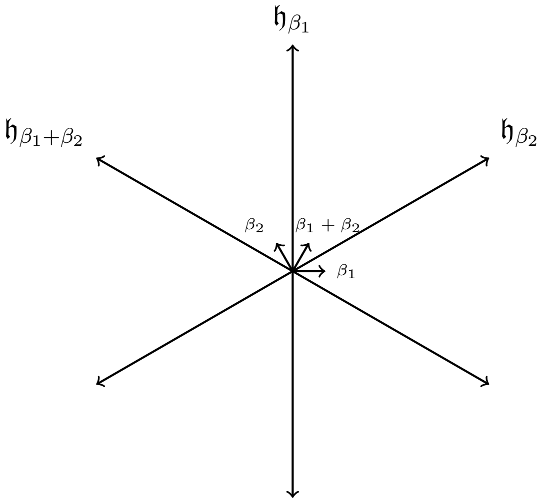
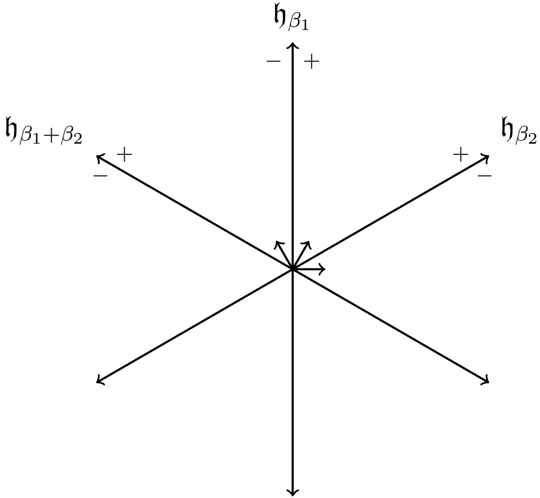
And so

$$\{\lambda + \ell\alpha \mid \ell \in \mathbb{Z}, V_{\lambda+\ell\alpha} \neq 0\} \qquad \text{(counting multiplicities)}$$

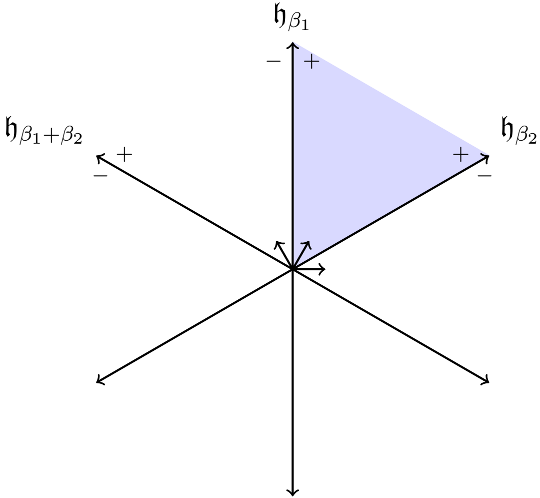
is a set of weights
(1) forming a string parallel to $\mathbb{R}\alpha$ and symmetric around $\mathfrak{h}_\alpha$, and
(2) whose distance from $\mathfrak{h}_\alpha$ are all in $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$.



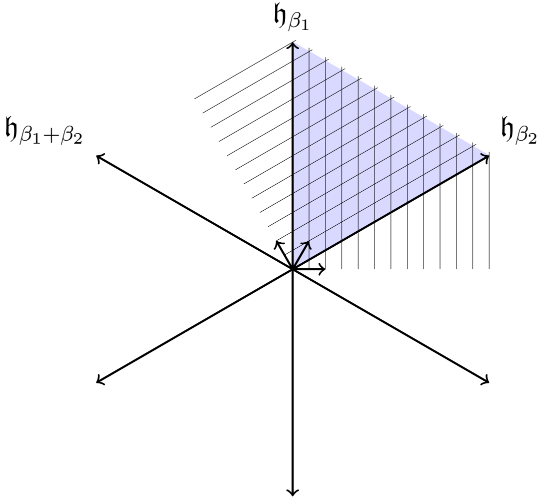
$\mathfrak{h}_{\beta_1}$
$\mathfrak{h}_{\beta_1+\beta_2}$
$\mathfrak{h}_{\beta_2}$
β_2
$\beta_1 + \beta_2$
β_1



$\mathfrak{h}_{\beta_1}$
$-$ $+$
$\mathfrak{h}_{\beta_1+\beta_2}$
$+$
$-$
$\mathfrak{h}_{\beta_2}$
$+$
$-$



$\mathfrak{h}_{\beta_1}$
$-$ $+$
$\mathfrak{h}_{\beta_1+\beta_2}$
$+$
$-$
$\mathfrak{h}_{\beta_2}$
$+$
$-$



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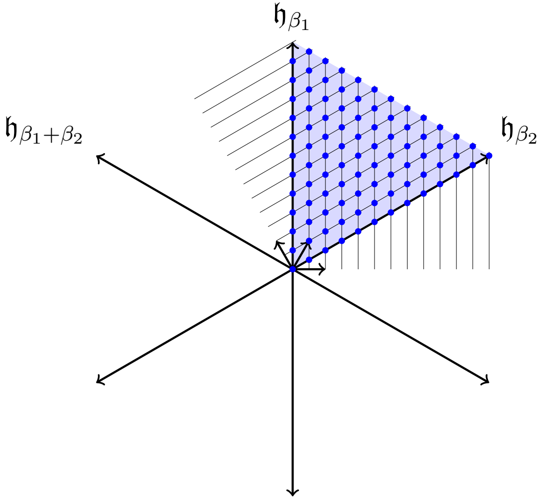
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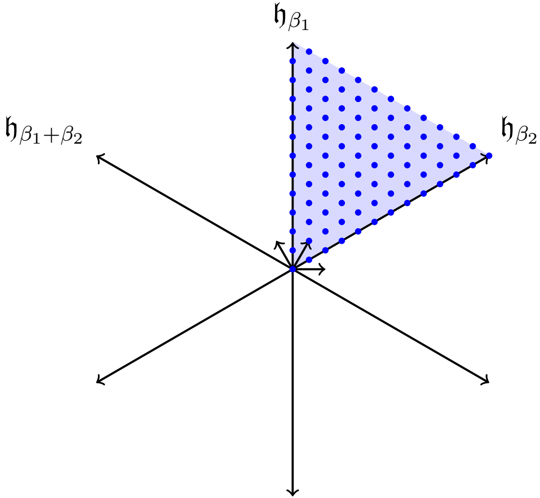
$\mathfrak{h}_{\beta_1+\beta_2}$
$\mathfrak{h}_{\beta_2}$



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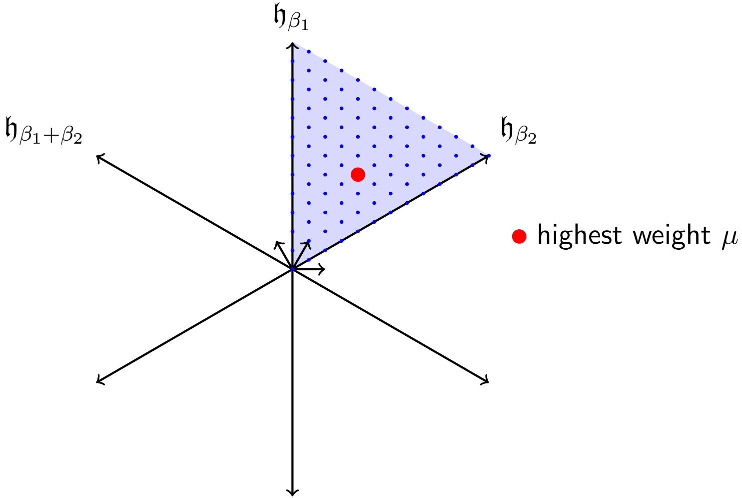
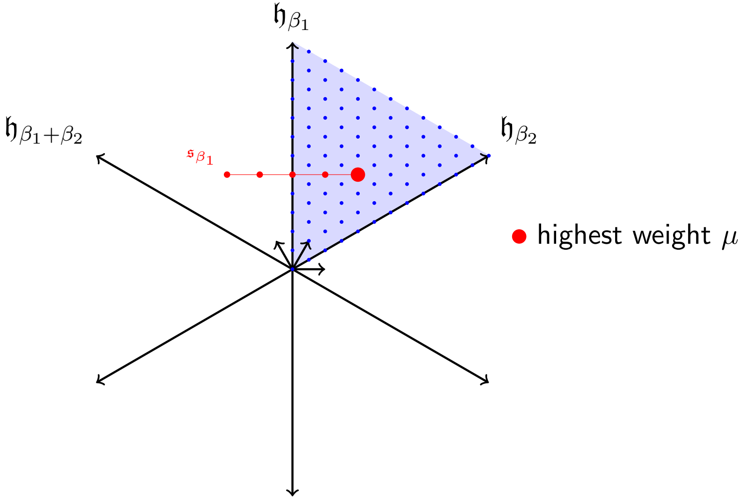
$\mathfrak{h}_{\beta_1+\beta_2}$
$\mathfrak{h}_{\beta_2}$



$\mathfrak{h}_{\beta_1}$
$\mathfrak{h}_{\beta_1+\beta_2}$
$\mathfrak{h}_{\beta_2}$
highest weight μ



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$\mathfrak{h}_{\beta_2}$
$\mathfrak{s}_{\beta_1}$
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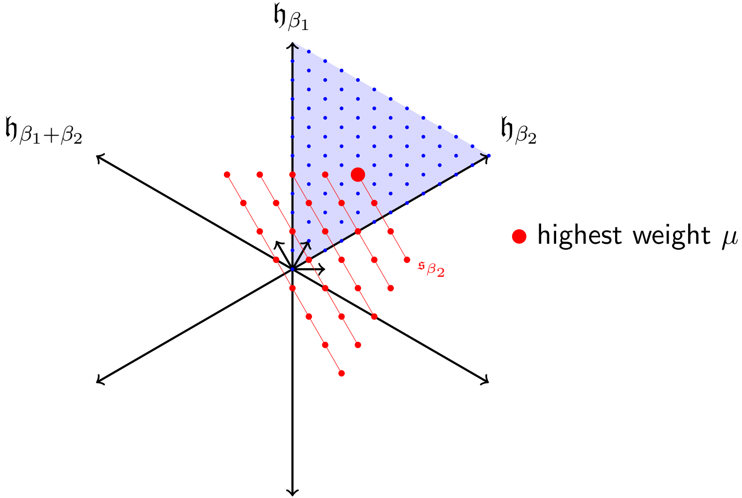

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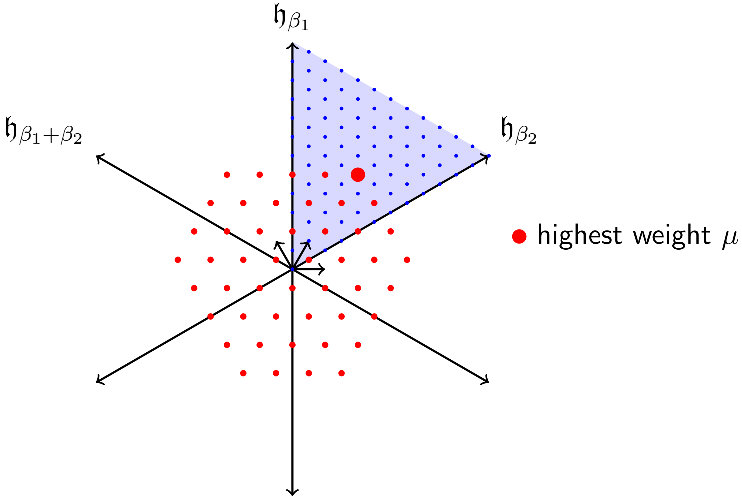
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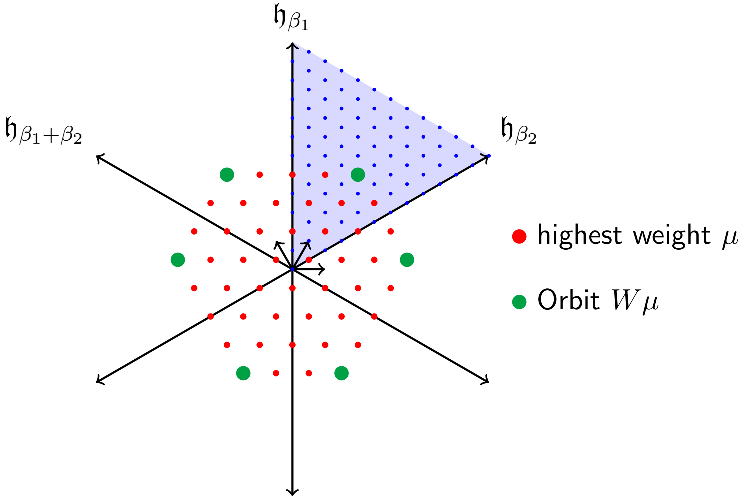

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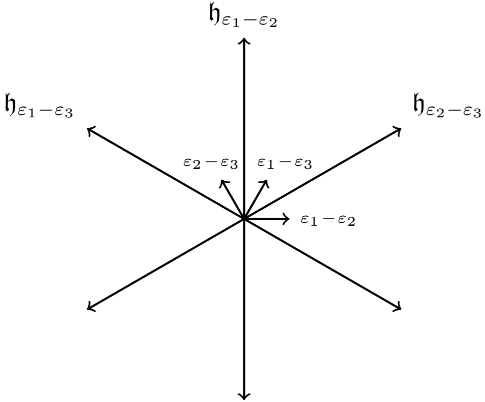
$\mathfrak{h}_{\beta_2}$
highest weight μ
Orbit $W\mu$

Aside about bases



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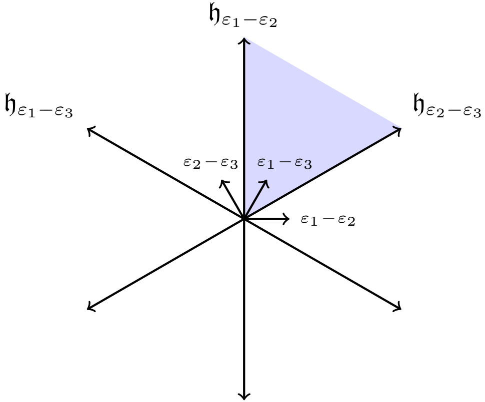

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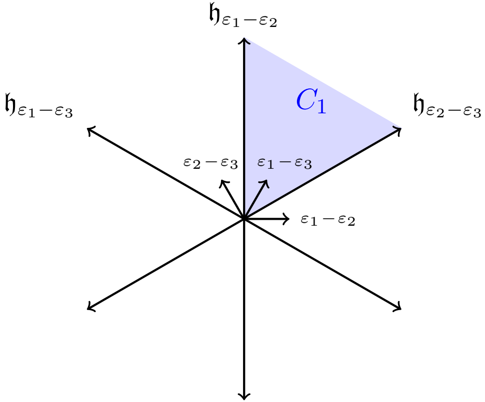

$C_1 : B = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$

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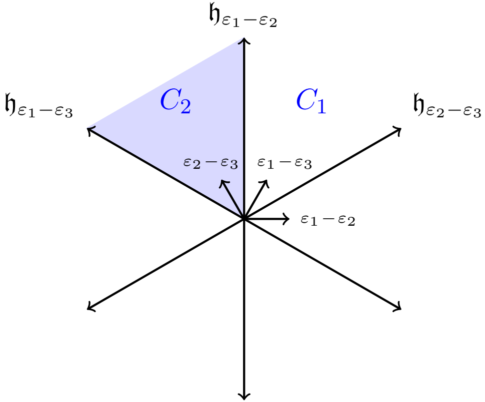

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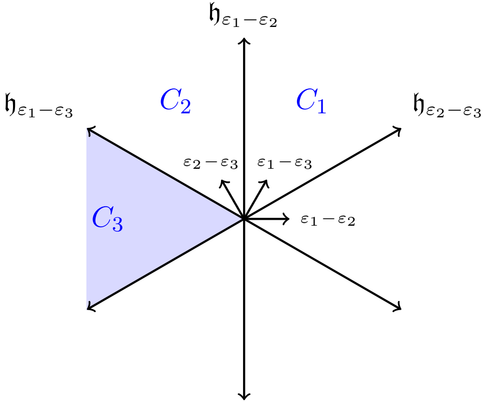

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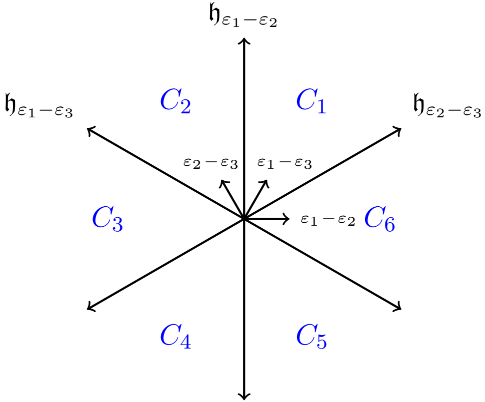

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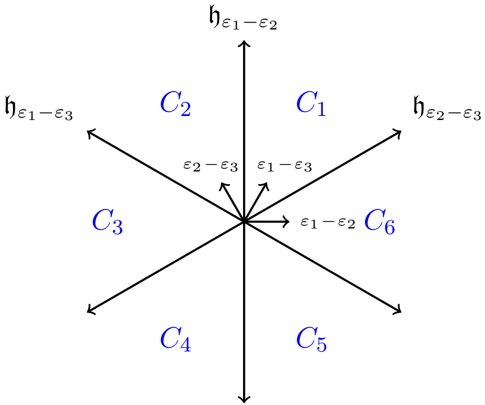
$C_5 : B = \{\varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_1\}$
$C_6 : B = \{\varepsilon_1 - \varepsilon_3, \varepsilon_3 - \varepsilon_2\}$

Aside about bases



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When we prove the existence of a base associated to a set of roots, we will see that to every chamber of $\mathfrak{h}^*_{\mathbb{R}}$, there is a base determined by the walls of that chamber.

Proposition

Let V be a highest weight module generated by primitive v^+ of weight μ.

(a) *If V is finite-dimensional, then $\langle \mu, \beta^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\beta \in B$.*

(b) *If $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}_{\geq} 0$, then for each $\alpha \in R^+$, as a $\mathfrak{s}_\alpha$-module, V is the sum of finite-dimensional $\mathfrak{s}_\alpha$-modules.*

(c) *The set of weights of V is invariant under the action of W. In particular, there is a bijection exchanging V_λ and $V_{s_\alpha(\lambda)}$, and so $\dim(V_\lambda) = \dim(V_{s_\alpha(\lambda)})$.*

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It only remains to show that the set of weights in V is finite. This amounts to the fact that there are only finitely many weights "less than" μ in C for which $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq} 0$. (Later)