Math 128: Lecture 13

April 21, 2014

So far:

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1. Finite-dimensional simple $\mathfrak{g}$-modules V are highest weight modules, i.e. there is some $v^+ \in V$ satisfying

   $hv^+ = \mu(h)v^+$ for some $\mu \in \mathfrak{h}^*$, and $h \in \mathfrak{h}$, and $\quad \mathfrak{n}^- v^+ = 0.$

2. Highest weight modules (of weight μ)
   (a) are simple,
   (b) are pairwise isomorphic if and only if they have the same weight, and
   (c) have top dimension ($\dim(V_\mu)$) equal to 1.

We also know that if $V_\lambda \neq 0$, then

$$\lambda = \mu - \sum_i \ell_i \beta_i \qquad \ell_i \in \mathbb{Z}_{\geq 0}, \beta_i \in B,$$

with for $\alpha_i \in R$, $x_i \in \mathfrak{g}_{\alpha_i}$, and $h \in \mathfrak{h}$,

$$h(x_1 \cdots x_m v^+) = \left(\mu + \sum_{i=1}^m \alpha_i\right)(h)(x_1 \cdots x_m v^+). \qquad (*)$$

Recall, to each $\alpha \in R^+$, there is a $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$ given by

$$\mathfrak{s}_\alpha = \left\langle x_\alpha, y_\alpha, h_{\alpha^\vee} \mid x_\alpha \in \mathfrak{g}_\alpha, y_\alpha \in \mathfrak{g}_{-\alpha}, h_{\alpha^\vee} = \frac{2}{\langle \alpha, \alpha \rangle} h_\alpha \right\rangle \subseteq \mathfrak{g},$$

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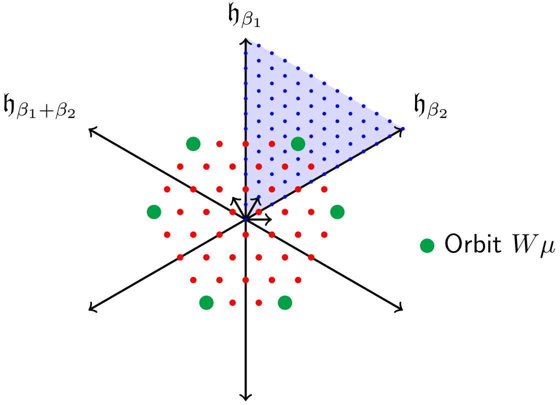
So if V is finite-dimensional of weight μ, its weight spaces look like



Proposition

Let V be a highest weight module generated by primitive v^+ of weight μ.

(a) *If V is finite-dimensional, then $\langle \mu, \beta^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\beta \in B$. And if $\langle \mu, \beta^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\beta \in B$, then $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in R^+$.*

(b) *If $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}_{\geq}0$ for each $\alpha \in R^+$, as a $\mathfrak{s}_\alpha$-module, V is the sum of finite-dimensional $\mathfrak{s}_\alpha$-modules.*

(c) *The set of weights of V is invariant under the action of W. In particular, there is a bijection exchanging V_λ and $V_{s_\alpha(\lambda)}$, and so $\dim(V_\lambda) = \dim(V_{s_\alpha(\lambda)})$.*

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Since the weights of V are given by $\lambda = \mu - \sum_i \ell_i \beta_i$ with $\ell_i \in \mathbb{Z}_{\geq 0}, \beta_i \in B$, part (a) says $\langle \lambda, \beta^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\beta \in B$ and all weights of V.

To show that a highest weight module is finite dimensional if and only if its highest weight is dominant integral, it only remains to show that the set of weights in V is finite.

This amounts to the fact that the W-orbit of the dominant integral weights that are “less than” μ is finite, and the weights of V are contained in that set. We will just have to show W is finite. (Later)

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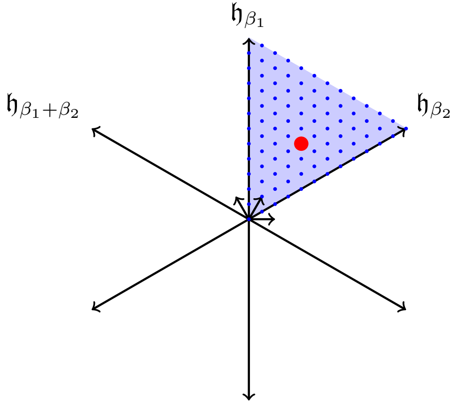

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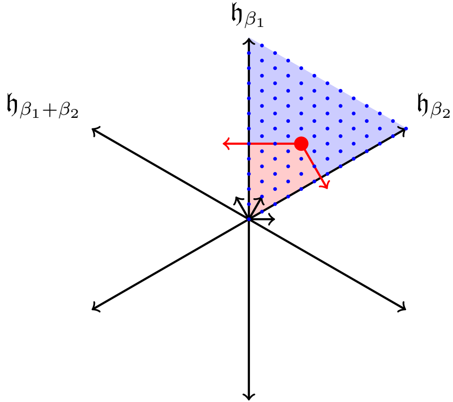

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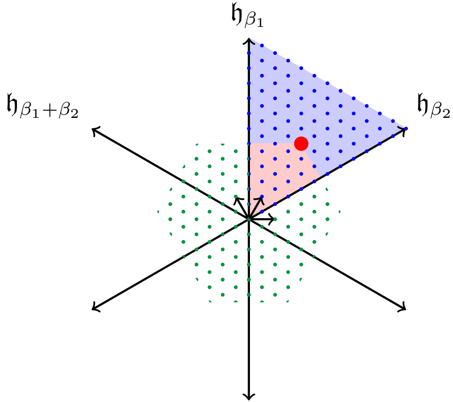

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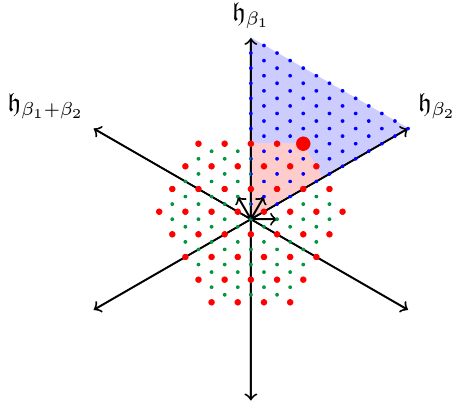

Generators of dominant integral weights

The dominant integral weights $P^+ \subseteq \mathfrak{h}^*_{\mathbb{R}}$ are those μ which satisfy

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But

$$\mathrm{proj}_\beta(\mu) = \frac{\langle \beta, \mu \rangle}{\langle \beta, \beta \rangle}\beta = \tfrac{1}{2}\langle \mu, \beta^\vee \rangle \beta,$$

so $\mu \in P^+$ if and only if

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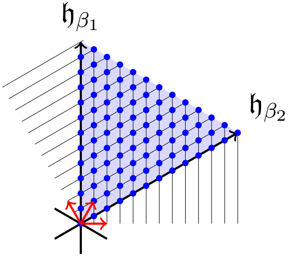

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So find $\Omega = \{\omega_1, \dots, \omega_r\}$ so that

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Then $P^+ = \mathbb{Z}_{\geq 0}\Omega$.
We call the weights in Ω the *fundamental weights*.

Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra with roots R. Let $B = \{\beta_1, \ldots, \beta_r\}$ be a base for R and let $\Omega = \{\omega_1, \ldots, \omega_r\}$ be the corresponding fundamental weights.

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Some notation:

$P^+ = \mathbb{Z}_{\geq 0}\Omega$	is the set of *dominant integral weights*
$P^{++} = \mathbb{Z}_{>0}\Omega$	is the set of *strongly dominant integral weights*
$P = \mathbb{Z}\Omega$	is the set of *integral weights*
B	is the set of *simple roots*
$R^\vee = \{\alpha^\vee \mid \alpha \in R\}$	is the set of *co-roots*
$B^\vee = \{\beta^\vee \mid \beta \in R\}$	is the set of *simple co-roots*

We say the fundamental weights are dual to the simple co-roots.