Existence of bases

A weight $\gamma \in h^*_\mathbb{R}$ is regular if $\gamma \notin \bigcup_{\alpha \in R} h_\alpha$. 

Let $R_+(\gamma) = \{ \alpha \in R | \langle \alpha, \gamma \rangle > 0 \}$. 

A root $\alpha \in R_+(\gamma)$ is decomposable if $\alpha = \alpha_1 + \alpha_2$ for some $\alpha_1, \alpha_2 \in R_+(\gamma)$, and that it is indecomposable otherwise. 

Let $B(\gamma) \subseteq R_+(\gamma)$ be the set of indecomposable roots in $R_+(\gamma)$. 

$\gamma$
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A weight $\gamma \in \mathfrak{h}_R^*$ is *regular* if $\gamma \notin \bigcup_{\alpha \in R} \mathfrak{h}_\alpha$. 

Let $\mathbb{R}_+ (\gamma) = \{ \alpha \in \mathfrak{r} | \langle \alpha, \gamma \rangle > 0 \}$. A root $\alpha \in \mathbb{R}_+ (\gamma)$ is *decomposable* if $\alpha = \alpha_1 + \alpha_2$ for some $\alpha_1, \alpha_2 \in \mathbb{R}_+ (\gamma)$, and that it is *indecomposable* otherwise.

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Let \( B(\gamma) \subseteq R^+(\gamma) \) be the set of indecomposable roots in \( R^+(\gamma) \).
Fix a fundamental chamber $C$, and therefore a base $B$ and positive set of roots $R^+$. With $B = \{\beta_1, \ldots, \beta_r\}$, let $s_i = s_{\beta_i}$. Let $W$ be the group generated by $\{s_\alpha \mid \alpha \in R\}$.
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**Lemma**

1. *The Weyl group $W$ is finite.*

2. *The form $\langle , \rangle$ on $\mathfrak{h}_R^*$ is $W$-invariant, i.e.*

   $$\langle w(\alpha), \beta \rangle = \langle \alpha, w^{-1}(\beta) \rangle, \quad \text{for all } \alpha, \beta \in R, w \in W.$$ 

3. *For all $\alpha \in R$, $w \in W$, we have $ws_{\alpha}w^{-1} = s_{w(\alpha)}$. Also, $w(\alpha^\vee) = w(\alpha)^\vee$.*

4. *The reflection associated to a simple root $\beta$ setwise fixes $R^+ - \{\beta\}$ and $R^- - \{-\beta\}$.*

5. *If $w = s_{i_1} s_{i_2} \cdots s_{i_{\ell - 1}}$ sends $\beta_{i_\ell}$ to a negative root, then $\omega s_{i_\ell} = s_{i_1} \cdots s_{i_{m - 1}} s_{i_{m + 1}} \cdots s_{i_{\ell - 1}}$ for some $1 \leq m < \ell$.*

6. *If $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ with $\ell$ minimal, then $w(\beta_{i_\ell}) < 0$.***