Math 128: Lecture 17

May 5, 2014

Last time:

Fix a base $B = \{\beta_i, \ldots, \beta_r\}$ and a fund. chamber $C = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda, \beta_i \rangle > 0\}$.
Let $s_i = s_{\beta_i}$ and $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$.
We saw $s_i \rho = \rho - \beta_i$ and $\rho = \sum_{i=1}^r \omega_i \in P^{++}$.

Theorem

1. *W acts transitively on Weyl chambers.*
2. *Fix a base B. For all $\alpha \in R$ there is some $w \in W$ with $w(\alpha) \in B$.*
3. *For any base B, W is generated by simple reflections (reflections associated to simple roots).*

   We showed for all $\alpha \in R$, we have $s_\alpha = w s_\beta w^{-1}$ with $\beta \in B, w \in \langle s_\gamma \mid \gamma \in B \rangle$
4. *W acts simply transitively on bases B of R.*

More on W

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1. W has a unique longest word w_0 which sends ρ to $-\rho$, so that $w_0 C$ is the unique Weyl chamber on the negative side of all hyperplanes.
2. The map $\det : W \to \{\pm 1\}$ defined by

$$w \mapsto \begin{cases} 1 & \text{if } w \text{ is the product of an even number of reflections,} \\ -1 & \text{if } w \text{ is the product of an odd number of reflections,} \end{cases}$$

   is well-defined (and equal to $(-1)^{\ell(w)}$). This is called the *alternating representation* or *sign representation* of W, and is sometimes also written as $\varepsilon(w)$.

Back to representation theory

Recall some things we know about representations of $\mathfrak{g}$:

1. For every $\lambda \in \mathfrak{h}^*$, there's a highest weight representation

$$L(\lambda) = U\mathfrak{g} \otimes_{U\mathfrak{b}} v_\lambda^+ \quad \text{where} \quad \begin{array}{ll} xv_\lambda^+ = 0 & \text{for all } x \in U^+ = U\mathfrak{n}^+, \\ hv_\lambda^+ = \lambda(h)v_\lambda^+ & \text{for all } h \in U^0 = U\mathfrak{h}. \end{array}$$

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2. With $R^+ = \{\alpha_1, \ldots, \alpha_m\}$ and $y_i \in \mathfrak{g}_{-\alpha_i}$, $L(\lambda)$ is spanned by weight vectors

$$y_1^{\ell_1} \cdots y_m^{\ell_m} v_\lambda^+ \quad \text{with weight} \quad \lambda - \sum_{i=1}^m \ell_m \alpha_m.$$

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$$\omega_i \text{ is determined by } \langle \omega_i, \beta_j^\vee \rangle = \delta_{ij}.$$

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$$m_\lambda = 1, \quad \text{and } m_\mu = m_{w\mu} \quad \text{for all} \quad w \in W.$$

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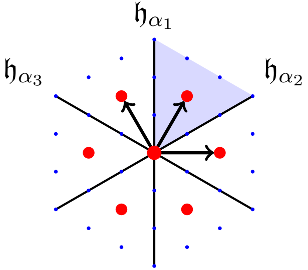
5. The set $P_\lambda = \{\mu \in \mathfrak{h}^* \mid \dim(L(\lambda)_\mu) > 0\}$ is the set of weights congruent to λ modulo R within the convex hull of $W\lambda$ in $\mathfrak{h}_{\mathbb{R}}^*$.

Example

Let $\mathfrak{g} = A_2$ have base $B = \{\beta_1, \beta_2 \mid \beta_i = \varepsilon_i - \varepsilon_{i+1}\}$, so that
$R^+ = \{\alpha_1 = \beta_1, \alpha_2 = \beta_2, \alpha_3 = \beta_1 + \beta_2\}$.
With $\lambda = \alpha_3$, the set P_λ is the red points in



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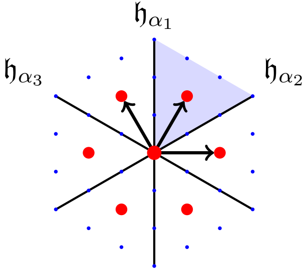

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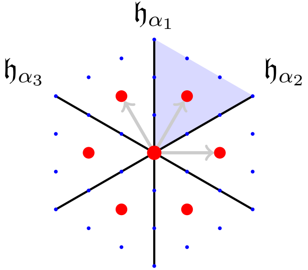

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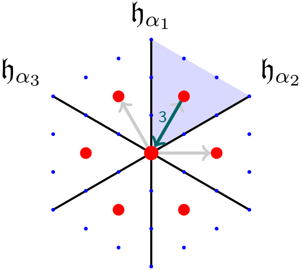

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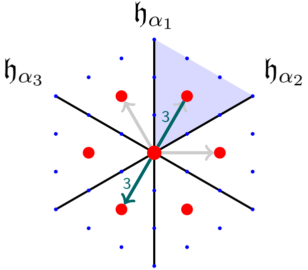

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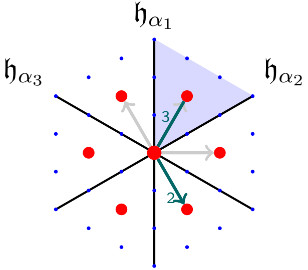

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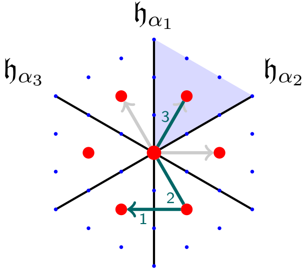

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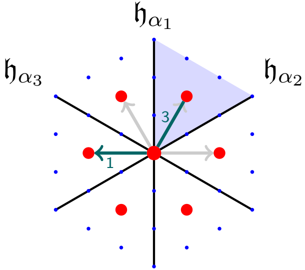

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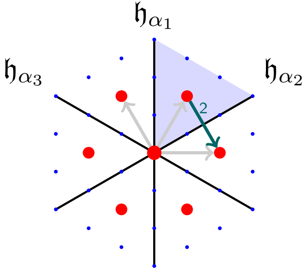

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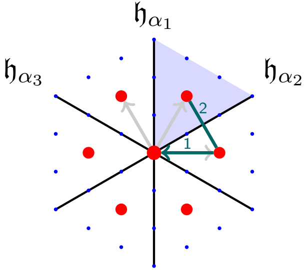

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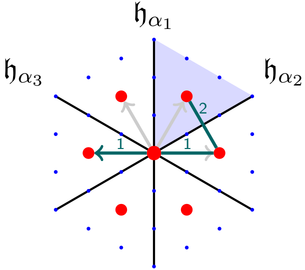

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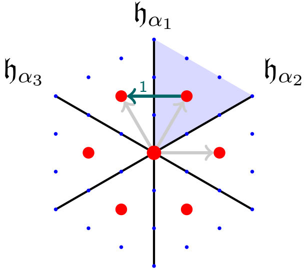

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Casimir element and Freudenthal's multiplicity formula

If $\{b_i\}$ is a basis of $\mathfrak{g}$, then there is a unique dual basis $\{b_i^*\}$ of $\mathfrak{g}$ determined by $\langle b_i, b_i^* \rangle = \delta_{ij}$.The *Casimir element* is

$$\kappa = \sum_{b_i} b_i b_i^* \in U\mathfrak{g}$$

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1. *κ does not depend on the choice of basis.*
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Theorem (Freudenthal's multiplicity formula)

Let m_μ be the dimension of $L(\lambda)_\mu$ in $L(\lambda)$, with $\lambda \in P^+$. Then m_μ is determined recursively by

$$m_\mu = \frac{2}{\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle} \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} \langle \mu + i\alpha, \alpha \rangle m_{\mu + i\alpha}.$$