Math 128: Lecture 17

May 5, 2014
Last time:
Fix a base $B = \{\beta_1, \ldots, \beta_r\}$ and a fund. chamber $C = \{\lambda \in \mathfrak{h}_\mathbb{R}^* \mid \langle \lambda, \beta_i \rangle > 0\}$.
Let $s_i = s_{\beta_i}$ and $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$.
We saw $s_i \rho = \rho - \beta_i$ and $\rho = \sum_{i=1}^r \omega_i \in P^{++}$.

Theorem

1. $W$ acts transitively on Weyl chambers.

2. Fix a base $B$. For all $\alpha \in R$ there is some $w \in W$ with $w(\alpha) \in B$.

3. For any base $B$, $W$ is generated by simple reflections (reflections associated to simple roots).
   
   We showed for all $\alpha \in R$, we have $s_\alpha = ws_\beta w^{-1}$ with $\beta \in B, w \in \langle s_\gamma \mid \gamma \in B \rangle$

4. $W$ acts simply transitively on bases $B$ of $R$. 
More on $W$

Fix a base $B = \{\beta_i, \ldots, \beta_r\}$ and a fund. chamber $C = \{\lambda \in h^*_R \mid \langle \lambda, \beta_i \rangle > 0\}$.

Let $s_i = s_{\beta_i}$ and $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$.

We saw $s_i \rho = \rho - \beta_i$ and $\rho = \sum_{i=1}^{r} \omega_i \in P^{++}$.

Define the length of an element $w \in W$, written $\ell(w)$ as the length of a minimal word in simple reflections generating $w$. 


More on $W$

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Define the \textit{length} of an element $w \in W$, written $\ell(w)$ as the length of a minimal word in simple reflections generating $w$.

Other facts:

1. $W$ has a unique longest word $w_0$ which sends $\rho$ to $-\rho$, so that $w_0C$ is the unique Weyl chamber on the negative side of all hyperplanes.
More on \( W \)

Fix a base \( B = \{\beta_i, \ldots, \beta_r\} \) and a fund. chamber \( C = \{\lambda \in \mathfrak{h}_\mathbb{R}^* \mid \langle \lambda, \beta_i \rangle > 0\} \).

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Define the length of an element \( w \in W \), written \( \ell(w) \) as the length of a minimal word in simple reflections generating \( w \).

Other facts:

1. \( W \) has a unique longest word \( w_0 \) which sends \( \rho \) to \(-\rho\), so that \( w_0 C \) is the unique Weyl chamber on the negative side of all hyperplanes.

2. The map \( \text{det} : W \rightarrow \{\pm 1\} \) defined by

\[
\begin{align*}
  w \mapsto \begin{cases} 
    1 & \text{if } w \text{ is the product of an even number of reflections}, \\
    -1 & \text{if } w \text{ is the product of an odd number of reflections},
  \end{cases}
\end{align*}
\]

is well-defined (and equal to \((-1)^{\ell(w)}\)). This is called the alternating representation or sign representation of \( W \), and is sometimes also written as \( \varepsilon(w) \).
Back to representation theory

Recall some things we know about representations of $\mathfrak{g}$:

1. For every $\lambda \in \mathfrak{h}^*$, there's a highest weight representation

$$L(\lambda) = U \mathfrak{g} \otimes_{U \mathfrak{b}} v_\lambda^+$$

where

$$x v_\lambda^+ = 0 \quad \text{for all } x \in U^+ = U \mathfrak{n}^+,$$

$$h v_\lambda^+ = \lambda(h) v_\lambda^+ \quad \text{for all } h \in U^0 = U \mathfrak{h}.$$
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2. With $R^+ = \{\alpha_1, \ldots, \alpha_m\}$ and $y_i \in \mathfrak{g}_{-\alpha_i}$, $L(\lambda)$ is spanned by weight vectors

$$y_1^{\ell_1} \cdots y_m^{\ell_m} v_+^\lambda \quad \text{with weight} \quad \lambda - \sum_{i=1}^m \ell_m \alpha_m.$$
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   \[ hv^+_\lambda = \lambda(h)v^+_\lambda \quad \text{for all} \quad h \in U^0 = U \mathfrak{h}. \]

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3. $L(\lambda)$ is finite-dimensional if and only if $\lambda \in P^+ = \sum_{i=1}^r \omega_i$, where

   $\omega_i$ is determined by $\langle \omega_i, \beta^\vee_j \rangle = \delta_{ij}$. 

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4. If $L(\lambda)$ is finite-dimensional, then with $m_\mu = \dim(L(\lambda)_\mu)$, we have

   $$m_\lambda = 1, \quad \text{and } m_\mu = m_{w\mu} \quad \text{for all } w \in W.$$
Back to representation theory

Recall some things we know about representations of $\mathfrak{g}$:

1. For every $\lambda \in \mathfrak{h}^*$, there’s a highest weight representation

$$L(\lambda) = U_{\mathfrak{g}} \otimes_{U_{\mathfrak{b}}} v^+_{\lambda} \text{ where } \begin{align*} x v^+_{\lambda} &= 0 \quad \text{for all } x \in U^+ = U_{n^+}, \\
 h v^+_{\lambda} &= \lambda(h) v^+_{\lambda} \quad \text{for all } h \in U^0 = U_{\mathfrak{h}}. \end{align*}$$

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$$m_\lambda = 1, \quad \text{and } m_\mu = m_{w\mu} \quad \text{for all } w \in W.$$ 

5. The set $P_\lambda = \{\mu \in \mathfrak{h}^* \mid \dim(L(\lambda)_\mu) > 0\}$ is the set of weights congruent to $\lambda$ modulo $R$ within the convex hull of $W \lambda$ in $\mathfrak{h}_\mathbb{R}^*$. 
Example

Let \( g = A_2 \) have base \( B = \{ \beta_1, \beta_2 \mid \beta_i = \varepsilon_i - \varepsilon_{i+1} \} \), so that \( R^+ = \{ \alpha_1 = \beta_1, \alpha_2 = \beta_2, \alpha_3 = \beta_1 + \beta_2 \} \).

With \( \lambda = \alpha_3 \), the set \( P_\lambda \) is the red points in

\[
\begin{align*}
\mathcal{h}_{\alpha_1} & \quad \mathcal{h}_{\alpha_2} \\
\mathcal{h}_{\alpha_3} & \\
\end{align*}
\]

so that \( P_{\alpha_3} = W_{\alpha_3} \sqcup \{0\} \) with \( W_{\alpha_3} = R \).
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![Diagram]

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What is $m_0$??
Casimir element and Freudenthal’s multiplicity formula

If \( \{b_i\} \) is a basis of \( g \), then there is a unique dual basis \( \{b_i^*\} \) of \( g \) determined by \( \langle b_i, b_i^* \rangle = \delta_{ij} \). The Casimir element is

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\kappa = \sum_{b_i} b_i b_i^* \in Ug
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where the sum is over the basis \( \{b_i\} \) and the dual basis \( \{b_i^*\} \).
Casimir element and Freudenthal’s multiplicity formula

If \{b_i\} is a basis of \mathfrak{g}, then there is a unique dual basis \{b_i^*\} of \mathfrak{g} determined by \langle b_i, b_i^* \rangle = \delta_{ij}. The Casimir element is

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**Theorem**

*Let \( \kappa \) be the Casimir element of \( \mathfrak{g} \).*

1. \( \kappa \) does not depend on the choice of basis.
2. \( \kappa \in \mathcal{Z}(U\mathfrak{g}), \) the center of \( U(\mathfrak{g}) \).
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**Theorem**

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2. \( \kappa \in Z(Ug) \), the center of \( U(g) \).

**Theorem (Freudenthal’s multiplicity formula)**

*Let \( m_\mu \) be the dimension of \( L(\lambda)_\mu \) in \( L(\lambda) \), with \( \lambda \in P^+ \). Then \( m_\mu \) is determined recursively by*

\[
m_\mu = \frac{2}{\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle} \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} \langle \mu + i\alpha, \alpha \rangle m_{\mu + i\alpha}.
\]