Math 128: Lecture 2

March 26, 2014

A (complex) *Lie algebra* is a vector space $\mathfrak{g}$ over $\mathbb{C}$ with a bracket $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying

(a) (*skew symmetry*) $[x, y] = -[y, x]$, and

(b) (*Jacobi identity*) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$,

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$\mathfrak{sp}_n(\mathbb{C}) = \{x \in \mathfrak{sl}_n \mid \langle xu, v\rangle + \langle u, xv\rangle = 0$ for all $u, v \in \mathbb{C}^n\}$,
where $\langle , \rangle$ is a skew-symmetric form on $\mathbb{C}^n$.

Classical Lie algebras

An algebra is *simple* if

(1) $\mathfrak{g}$ has no nontrivial proper ideals
(the only subspaces $\mathfrak{a} \subseteq \mathfrak{g}$ satisfying $[\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a}$ are $\mathfrak{g}$ and 0), and

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Four infinite families of simple Lie algebras, called the *classical Lie algebras*:

Type A_r: $\mathfrak{sl}_{r+1}(\mathbb{C})$, $r \geq 1$

Type B_r: $\mathfrak{so}_{2r+1}(\mathbb{C})$, $r \geq 2$

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The *exceptional Lie algebras*,

E_6, E_7, E_8, F_4, and G_2,

complete the list of simple complex Lie algebras.

Standard and adjoint representations

A *representation* of a Lie algebra is a vector space V together with a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{End}(V)$ satisfying $\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$.

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The *adjoint* representation of a Lie algebra $\mathfrak{g}$ is

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathrm{End}(\mathfrak{g}) \\ x &\mapsto \mathrm{ad}_x = [\cdot, x], \qquad \text{i.e. } \mathrm{ad}_x(y) = [y, x]. \end{aligned}$$

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Vector space: $\mathbb{C}$-span(free group on $\mathfrak{g}$-basis)
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$U\mathfrak{g}$ is called the *universal enveloping algebra* of $\mathfrak{g}$.