Math 128: Lecture 2

March 26, 2014
A (complex) Lie algebra is a vector space $\mathfrak{g}$ over $\mathbb{C}$ with a bracket
$[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying

(a) (skew symmetry) $[x, y] = -[y, x]$, and

(b) (Jacobi identity) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,$

for all $x, y, z \in \mathfrak{g}$. 
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**Favorite examples:**

$$\mathfrak{gl}_n(\mathbb{C}) = \text{End}(\mathbb{C}^n)$$
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- $\mathfrak{gl}_n(\mathbb{C}) = \text{End}(\mathbb{C}^n)$
- $\mathfrak{sl}_n(\mathbb{C}) = \{ x \in \mathfrak{gl}_n(\mathbb{C}) \mid \text{tr}(x) = 0 \}$
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$\mathfrak{sl}_n(\mathbb{C}) = \{x \in \mathfrak{gl}_n(\mathbb{C}) \mid \text{tr}(x) = 0\}$

$\mathfrak{so}_n(\mathbb{C}) = \{x \in \mathfrak{sl}_n \mid \langle xu, v \rangle + \langle u, xv \rangle = 0 \text{ for all } u, v \in \mathbb{C}^n\}$, where $\langle \cdot, \cdot \rangle$ is a symmetric form on $\mathbb{C}^n$. 
A (complex) **Lie algebra** is a vector space \( g \) over \( \mathbb{C} \) with a bracket \([,]: g \otimes g \to g\) satisfying

(a) (**skew symmetry**) \([x, y] = -[y, x]\), and

(b) (**Jacobi identity**) \([x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\), for all \( x, y, z \in g \).

**Favorite examples:**

\[
\begin{align*}
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\text{where } \langle , \rangle \text{ is a symmetric form on } \mathbb{C}^n. \\
\mathfrak{sp}_n(\mathbb{C}) &= \{ x \in \mathfrak{sl}_n \mid \langle xu, v \rangle + \langle u, xv \rangle = 0 \text{ for all } u, v \in \mathbb{C}^n \}, \\
\text{where } \langle , \rangle \text{ is a skew-symmetric form on } \mathbb{C}^n.
\end{align*}
\]
Classical Lie algebras

An algebra is *simple* if

(1) $g$ has no nontrivial proper ideals
    (the only subspaces $a \subseteq g$ satisfying $[a, g] \subseteq a$ are $g$ and 0),
    and

(2) $g$ is not abelian ($[g, g] \neq 0$).

Four infinite families of simple Lie algebras, called the classical Lie algebras:

Type $A_r$: $\mathfrak{sl}_{r+1}(\mathbb{C})$, $r \geq 1$

Type $B_r$: $\mathfrak{so}_{2r+1}(\mathbb{C})$, $r \geq 2$

Type $C_r$: $\mathfrak{sp}_{2r}(\mathbb{C})$, $r \geq 3$

Type $D_r$: $\mathfrak{so}_{2r}(\mathbb{C})$, $r \geq 4$

The exceptional Lie algebras, $E_6$, $E_7$, $E_8$, $F_4$, and $G_2$, complete the list of simple complex Lie algebras.
Classical Lie algebras

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   and

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- **Type** \( A_r \): \( \mathfrak{sl}_{r+1}(\mathbb{C}) \), \( r \geq 1 \)
- **Type** \( B_r \): \( \mathfrak{so}_{2r+1}(\mathbb{C}) \), \( r \geq 2 \)
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- **Type** \( D_r \): \( \mathfrak{so}_{2r}(\mathbb{C}) \), \( r \geq 4 \)

The *exceptional Lie algebras*,

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E_6, E_7, E_8, F_4, \text{ and } G_2,
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complete the list of simple complex Lie algebras.
Standard and adjoint representations

A representation of a Lie algebra is a vector space $V$ together with a Lie algebra homomorphism $\rho : \mathfrak{g} \to \text{End}(V)$ satisfying

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x).$$
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We saw the standard representations of $\mathfrak{sl}_n$, $\mathfrak{so}_n$, and $\mathfrak{sp}_n$. 
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We saw the *standard* representations of $\mathfrak{sl}_n$, $\mathfrak{so}_n$, and $\mathfrak{sp}_n$.

The *adjoint* representation of a Lie algebra $\mathfrak{g}$ is

$$\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g})$$

$$x \mapsto \text{ad}_x = [\cdot, x], \quad \text{i.e. } \text{ad}_x(y) = [y, x].$$
Let $A$ be an algebra over $\mathbb{C}$.
Then let $L(A)$ be the Lie algebra with
Vector space: $A$
Bracket: $[x, y] = xy - yx$. 
Algebras ↔ Lie algebras

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Bracket: $[x, y] = xy - yx$.

Let $\mathfrak{g}$ be a complex Lie algebra. Then let $U\mathfrak{g}$ be the algebra with
Vector space: $\mathbb{C}$-span(free group on $\mathfrak{g}$-basis)
Multiplication: satisfies relation $xy - yx = [x, y]$
Algebras $\leftrightarrow$ Lie algebras

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$U\mathfrak{g}$ is called the universal enveloping algebra of $\mathfrak{g}$. 