Math 128: Lecture 20

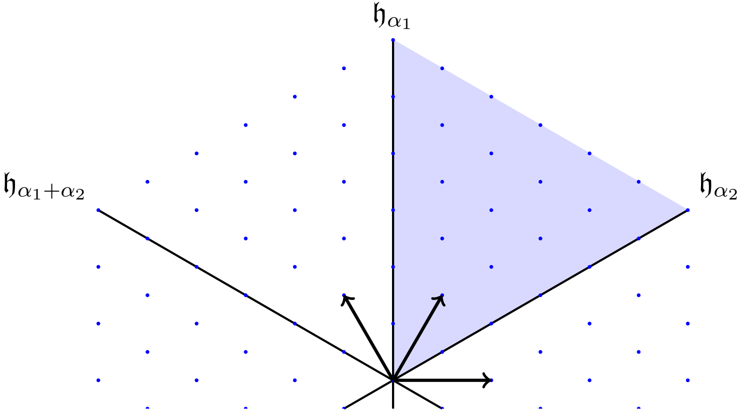
May 12, 2014

Weigh space multiplicities:

We're trying to calculate m_μ^λ, the dimension of $L(\lambda)_\mu$ in $L(\lambda)$, with $\lambda \in P^+ = \mathbb{Z}_{\geq 0}\{\omega_1, \ldots, \omega_r\}$.

1. First solution: Freudenthal's multiplicity formula.
$$m_\mu^\lambda = \frac{2}{\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle} \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} \langle \mu + i\alpha, \alpha \rangle m_{\mu + i\alpha}^\lambda.$$
2. Second solution: Weyl character formula. The character of a finite-dimensional $\mathfrak{g}$-module V is
$$\mathrm{ch}(V) = \sum_{\lambda \in P} \dim(V_\lambda) X^\lambda.$$
For irreducible modules, the character is given by
$$\mathrm{ch}(L(\lambda)) = \frac{a_{\lambda+\rho}}{a_\rho} \quad \text{where} \quad a_{\lambda+\rho} = \sum_{w \in W} \det(w) X^{w(\lambda+\rho)}.$$
3. Third solution: Path model.

Littelmann path model



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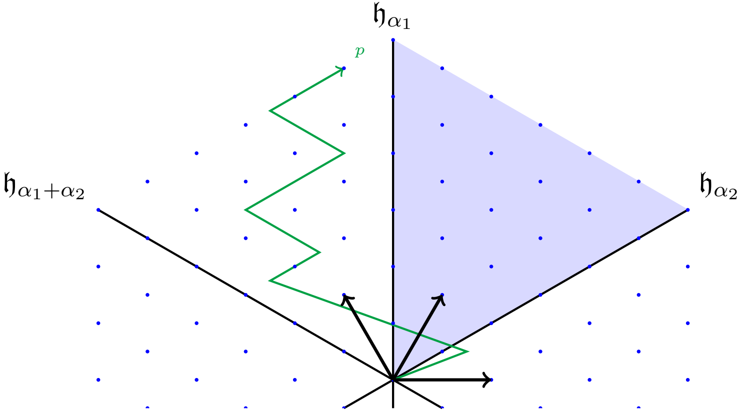

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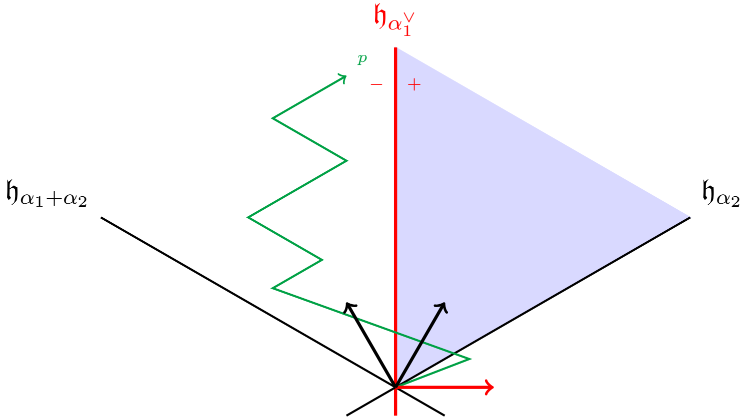

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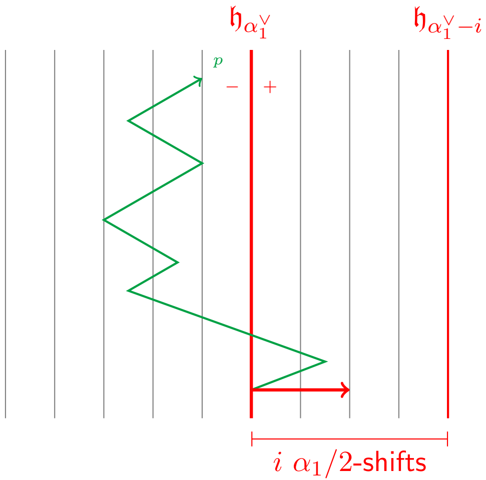

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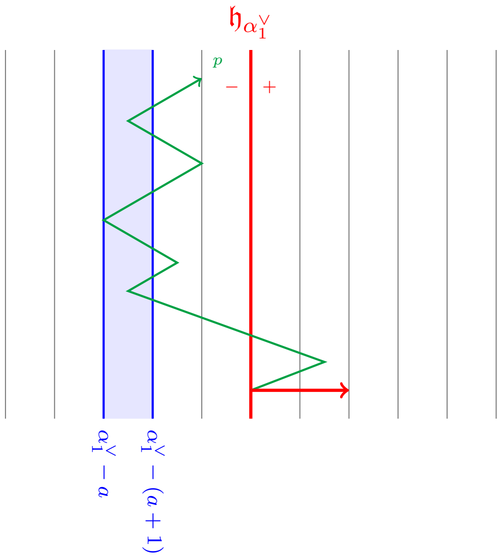

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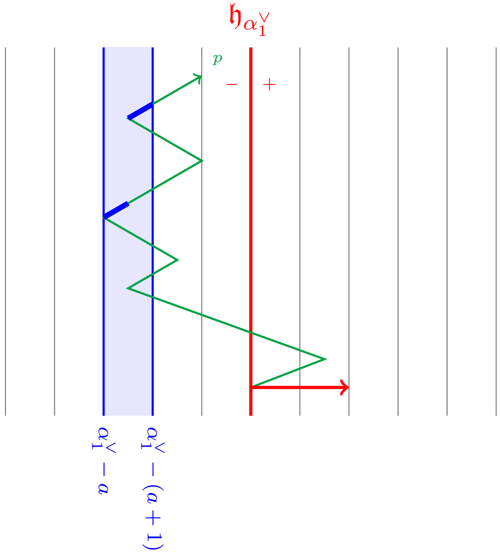

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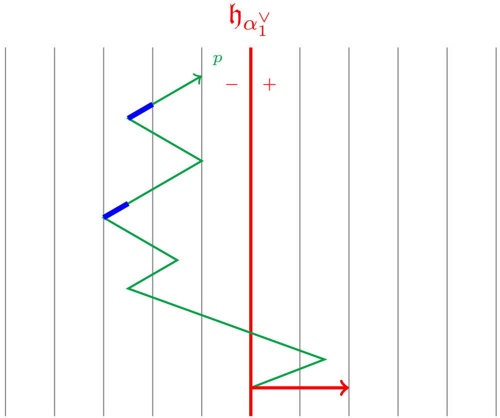

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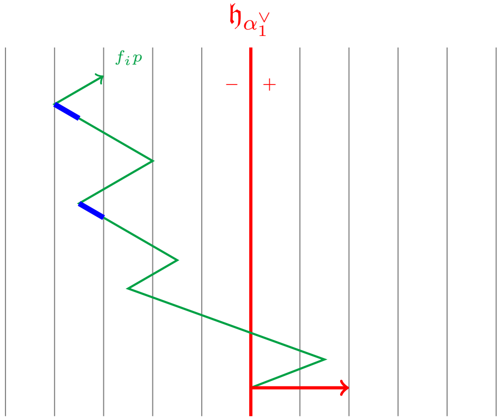

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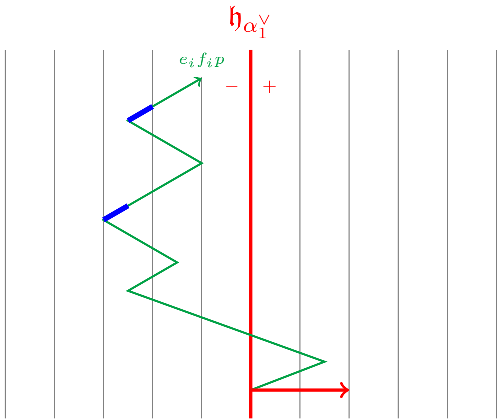

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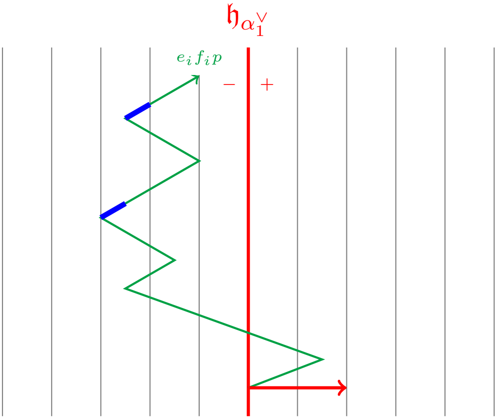
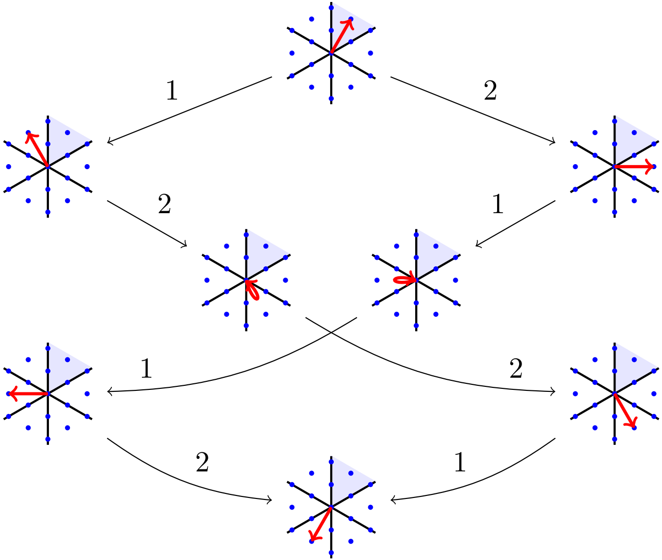


A *crystal* $\mathcal{B}$ is a set of paths closed under $\{f_i, e_i \mid i = 1, \ldots, r\}$.

The *crystal graph* has vertices $p \in \mathcal{B}$ and edges $p \xrightarrow{i} f_i p$.

Working example

Fix $\mathfrak{g} = A_2$ with base $B = \{\beta_1, \beta_2 \mid \beta_i = \varepsilon_i - \varepsilon_{i+1}\}$. Calculate m_0^ρ.



Highest weight crystals

A *highest weight path* is a path p satisfying $e_i p = 0$ for all i, which is equivalent to

$$p(1) \in P^+ \qquad \text{and} \qquad p(t) \in C - \rho \text{ for all } t \in [0,1].$$

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Prop. The crystals generated highest weight paths of the same weight are isomorphic.

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The *character of a crystal* is

$$\mathrm{ch}(\mathcal{B}) = \sum_{p \in \mathcal{B}} X^{\mathrm{wt}(p)}.$$

Theorem

For $\lambda \in P^+$, $\mathrm{ch}(\mathcal{B}(\lambda)) = \mathrm{ch}(L(\lambda))$.

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Proposition

Let $\mathcal{B}, \mathcal{B}'$ *be finite crystals.*

1. $\mathrm{ch}(\mathcal{B}) = \mathrm{ch}(\mathcal{B}')$ *if and only if* $\mathcal{B} \cong \mathcal{B}'$.
2. *The union* $\mathcal{B} \sqcup \mathcal{B}'$ *is a crystal, and*

$$\mathrm{ch}(\mathcal{B} \sqcup \mathcal{B}') = \mathrm{ch}(\mathcal{B}) + \mathrm{ch}(\mathcal{B}').$$

3. $\mathrm{ch}(\mathcal{B}) = \sum_{\substack{p \in \mathcal{B} \\ p \text{ is highest weight}}} \mathrm{ch}(\mathcal{B}(\mathrm{wt}(p))).$

Tensor product rules

The *concatenation* of two paths p, p' is defined by

$$pp' = \begin{cases} p(2t) & 0 \leq t \leq 1/2, \\ p(1) + p'(2(t - 1/2)) & 1/2 \leq t \leq 1. \end{cases}$$

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$$\mathcal{B}(V \otimes V') = \{pp' \mid p \in \mathcal{B}(V), p' \in \mathcal{B}(V')\}.$$

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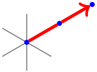
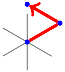
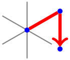
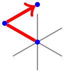
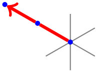
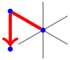
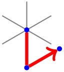
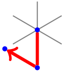
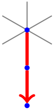
2. *With $\lambda, \mu \in P^+$, and p_λ^+ highest weight in $\mathcal{B}(\lambda)$,*

$$\mathrm{ch}(L(\lambda) \otimes L(\mu)) = \sum_{\substack{q \in \mathcal{B}(\mu) \\ p_\lambda^+ q \text{ highest weight}}} \mathrm{ch}(L(\lambda + \mathrm{wt}(q))).$$

$\mathcal{B}(L(\omega_1) \otimes L(\omega_1))$ is the set containing

$p_1^2 =$  , $p_1 p_2 =$  , $p_1 p_3 =$  ,

$p_2 p_1 =$  , $p_2^2 =$  , $p_2 p_3 =$  ,

$p_3 p_1 =$  , $p_3 p_2 =$  , $p_3^2 =$  .

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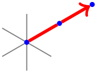
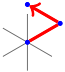
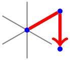
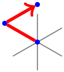
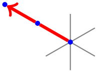
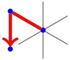
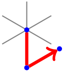
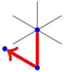
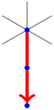
$\rightarrow$ $p_1^2 =$  , $p_1p_2 =$  , $p_1p_3 =$  , $\leftarrow$

$p_2p_1 =$  , $p_2^2 =$  , $p_2p_3 =$  ,

$p_3p_1 =$  , $p_3p_2 =$  , $p_3^2 =$  .

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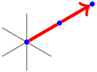
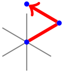
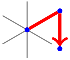
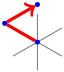
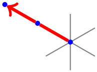
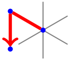
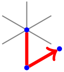
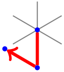
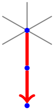
$\rightarrow$ $p_1^2 =$  , $p_1 p_2 =$  , $p_1 p_3 =$  , $\leftarrow$

$p_2 p_1 =$  , $p_2^2 =$  , $p_2 p_3 =$  ,

$p_3 p_1 =$  , $p_3 p_2 =$  , $p_3^2 =$  .

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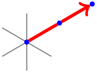
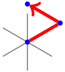
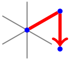
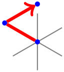
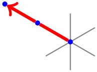
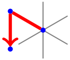
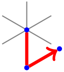
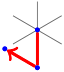
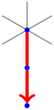
$\rightarrow$ $p_1^2 =$  , $p_1p_2 =$  , $p_1p_3 =$  , $\leftarrow$

$p_2p_1 =$  , $p_2^2 =$  , $p_2p_3 =$  ,

$p_3p_1 =$  , $p_3p_2 =$  , $p_3^2 =$  .

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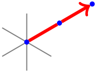
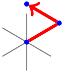
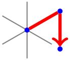
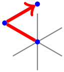
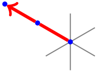
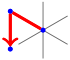
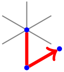
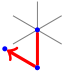
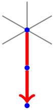
$\rightarrow$ $p_1^2 =$  , $p_1 p_2 =$  , $p_1 p_3 =$  , $\leftarrow$

$p_2 p_1 =$  , $p_2^2 =$  , $p_2 p_3 =$  ,

$p_3 p_1 =$  , $p_3 p_2 =$  , $p_3^2 =$  .

Highest weight paths:

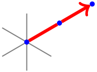
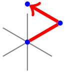
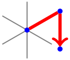
p_1^2 with weight $2\omega_1$

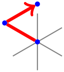
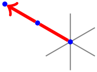
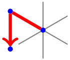
$p_1 p_2$ with weight ω_2

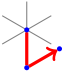
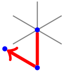
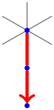
So $\mathrm{ch}(L(\omega_1) \otimes L(\omega_1)) = \mathrm{ch}(L(2\omega_1)) + \mathrm{ch}(L(\omega_2))$, implying

$$L(\omega_1) \otimes L(\omega_1) \cong L(2\omega_1) \oplus L(\omega_2)$$

$\mathcal{B}(L(\omega_1) \otimes L(\omega_1)) = \mathcal{B}(L(\square) \otimes L(\square))$ is the set containing

$\rightarrow$ $p_1^2 =$ , $p_1p_2 =$ , $p_1p_3 =$ , $\leftarrow$

$p_2p_1 =$ , $p_2^2 =$ , $p_2p_3 =$ ,

$p_3p_1 =$ , $p_3p_2 =$ , $p_3^2 =$ .

Highest weight paths:

$$p_1^2 \text{ with weight } 2\omega_1 \quad = \square\square$$

$$p_1p_2 \text{ with weight } \omega_2 \quad = \begin{smallmatrix}\square\\\square\end{smallmatrix}$$

So $\mathrm{ch}(L(\omega_1) \otimes L(\omega_1)) = \mathrm{ch}(L(2\omega_1)) + \mathrm{ch}(L(\omega_2))$, implying

$$L(\square) \otimes L(\square) = L(\omega_1) \otimes L(\omega_1) \cong L(2\omega_1) \oplus L(\omega_2) = L(\square\square) \oplus L\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)$$

Back to tableaux

$\mathcal{B}(\omega_1) = \{p_i \mid i = 1, \ldots, r+1\}$ where p_i is the straight-line path to $\varepsilon_i - \frac{1}{r+1}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{r+1})$.

$p_1 =$  $p_2 =$  and $p_3 =$  .

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Generate $\mathcal{B}(\lambda)$ with the path $p_\lambda^+ = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_r^{\lambda_r}$.

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Generate $\mathcal{B}(\lambda)$ with the path $p_\lambda^+ = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_r^{\lambda_r}$.

Is this ok?

If $p = p_{i_1} p_{i_2} \cdots p_{i_n}$, then $p(t) \in C - \rho \ \forall t$ iff every *initial path* $p_{i_1} \cdots p_{i_j}$ has weight

$$\mathrm{wt}(p_{i_1} \cdots p_{i_j}) = \sum_{k=1}^{j} (\omega_{i_k} - \omega_{i_k - 1}) \in P^+$$

(iff its weight is the sum of ω_i's).

Back to tableaux

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Back to tableaux

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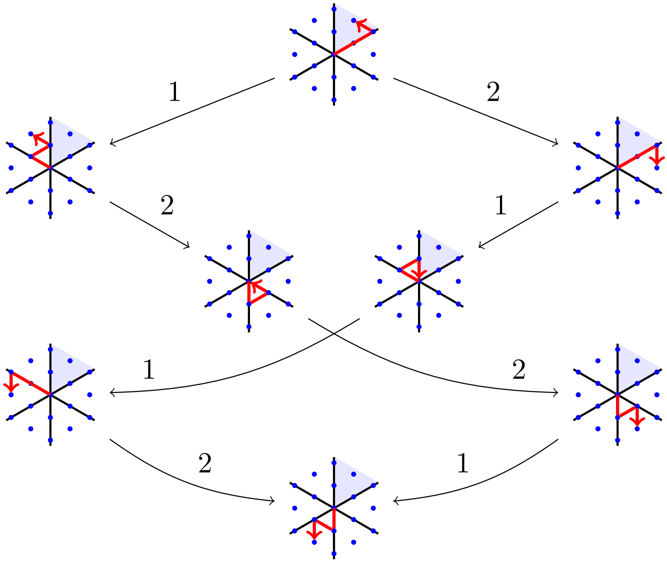
$$\mathrm{wt}(p_{i_1} \cdots p_{i_j}) = \sum_{k=1}^{j} (\omega_{i_k} - \omega_{i_k - 1}) \in P^+$$

(iff its weight is the sum of ω_i's).

Another way: define the *reading word* of p to be $i_1 i_2 \cdots i_n$.

Then $p(t) \in C - \rho \ \forall t$ if any only if every *initial subword* $i_1 i_2 \cdots i_j$ of the reading word of p has the property that is contains more 1's than 2's, more 2's than 3's, and so on.


1
2
2
1
1
2
2
1