Math 128: Lecture 22

May 16, 2014

Last time: Decomposing modules

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The canonical map

$$\mathrm{Hom}(A^\lambda, M) \otimes A^\lambda \to M \quad \text{defined by} \quad \phi \otimes u \mapsto \phi(u)$$

produces an isomorphism

$$\mathrm{Hom}(A^\lambda, M) \otimes A^\lambda \cong M^{(\lambda)}.$$

So, for one, $m_M(\lambda) = \dim(\mathrm{Hom}(A^\lambda, M))$.

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Define the *centralizer* of A (in $\mathrm{End}(M)$) to be

$$\mathrm{End}_A(M) = \{\phi \in \mathrm{End}(M) \mid a\phi(m) = \phi(a \cdot m) \text{ for all } a \in A, m \in M\},$$

i.e. $\mathrm{End}_A(M)$ is the set of all endomorphisms of M which commute with the action of A.

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Let $B = \mathrm{End}_A(M)$. M is not only a module for A and B individually, but since their actions commute, it's an A, B *bimodule*, i.e. a module for $A \otimes B$.

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Question: How does M decompose as an A, B bimodule?

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There is a natural action of $\mathrm{End}_A(M)$ on $\mathrm{Hom}(A^\lambda, M)$ by

$$b \cdot \phi : v \mapsto b \cdot \phi(v)$$

for all $b \in B$, $\phi \in \mathrm{Hom}(A^\lambda, M)$, and $v \in A^\lambda$.
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Theorem (Double centralizer theorem)

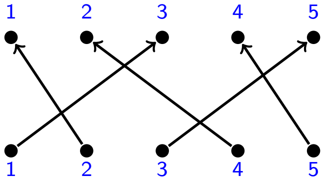
Let M be a vector space, and $A \subseteq \mathrm{End}(M)$. Then the algebra $B = \mathrm{End}_A(M)$ is semisimple, one has $\mathrm{End}_B(M) = A$, and M has the multiplicity-free complete decomposition

$$M \cong \bigoplus_{\widehat{M}} A^\lambda \otimes B^\lambda$$

as an (A, B)-bimodule, where $\{B^\lambda \mid \lambda \in \widehat{M}\}$ are distinct simple B-modules.

Example: The symmetric group and $\mathfrak{sl}_n$

The **symmetric group** S_k (permutations) as diagrams:



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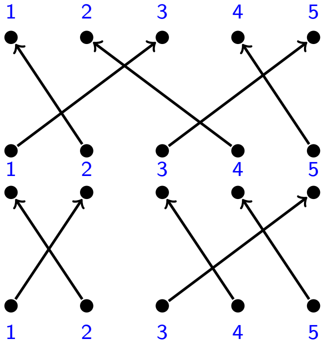

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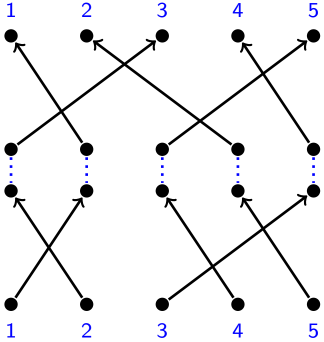

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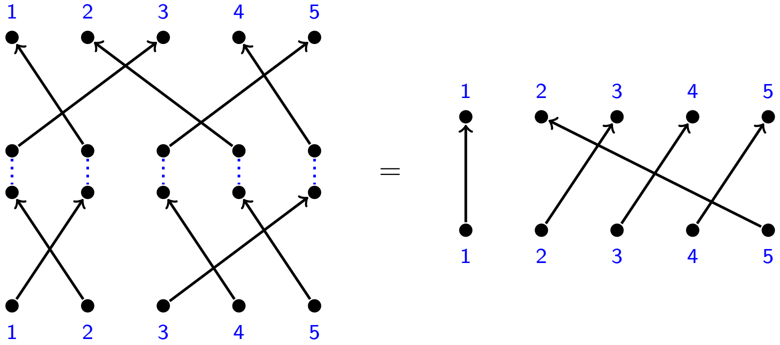

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$$x \cdot (v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j=1}^{k} v_{i_1} \otimes \cdots \otimes x v_2 \otimes \cdots \otimes v_{i_k}.$$

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S_k also acts on $V^{\otimes k}$ by place permutation:

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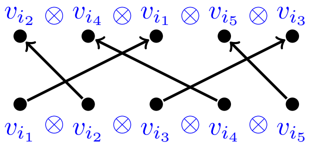

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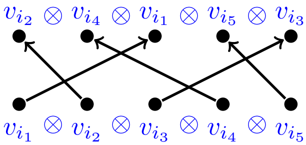

These actions commute!

Schur-Weyl Duality

Let A be one of

the group algebra $\mathbb{C}\mathrm{GL}_n(\mathbb{C})$	where $\mathrm{GL}_n(\mathbb{C}) = \{g \in M_n(\mathbb{C}) \mid \det(g) \neq 0\}$,
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with standard representation $V = \mathbb{C}^n$. Let $B = \mathbb{C}S_k$.
Schur (1901): A and B have commuting actions on $V^{\otimes k}$ with

$$\mathrm{End}_A\left(V^{\otimes k}\right) = \underbrace{\pi(B)}_{\substack{\text{(img of}\\ B\text{-action)}}} \quad \text{and} \quad \mathrm{End}_B\left(V^{\otimes k}\right) = \underbrace{\rho(A)}_{\substack{\text{(img of}\\ A\text{-action)}}}$$

and this double-centralizer relationship produces

$$V^{\otimes k} \cong \bigoplus_{\lambda \vdash k} L(\lambda) \otimes S^\lambda \quad \text{as a } A\text{-}B \text{ bimodule.}$$