Math 128: Lecture 23

May 19, 2014

More on Centralizers

The double centralizer theorem says that for a vector space M, $A \subseteq \mathrm{End}(M)$ semisimple, and $B = \mathrm{End}_A(M)$, we have

1. B is semisimple;
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Actually,

$$Z(A) = \mathrm{End}_A(M) \cap A = \mathrm{End}_A(M) \cap \mathrm{End}_B(M) = B \cap \mathrm{End}_B(M) = Z(B).$$

So the centers of both algebras are generated by the same centrally primitive idempotents, which were the elements of $Z(A)$ satisfying

$$p_\lambda^2 = p_\lambda, \qquad p_\lambda p_\mu = p_\mu p_\lambda = 0 \text{ for } \lambda \neq 0, \quad \text{and} \quad \sum_{\lambda \in \widehat{A}} p_\lambda = 1,$$

so that

$$Z(A) = \mathbb{C}\{p_\lambda \mid \lambda \in \widehat{A}\} \quad \text{and} \quad p_\lambda M = M^{(\lambda)}.$$

Computing idempotents [GP, §7]

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Suppose that the trace form $\langle a, b \rangle = \mathrm{tr}(ab)$ on the *regular representation* is nondegenerate.

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Then

$$p_\lambda = \frac{1}{c_\lambda} \sum_{b \in \mathcal{B}} \chi^\lambda(b^*) b.$$

The Temperley-Lieb algebra $TL_k(x)$ is generated over $\mathbb{C}$ by $e_1, \ldots, e_{k-1}$ with relations

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The Brauer algebra $B_k(x)$ is generated over $\mathbb{C}$ by $\mathbb{C}S_k = \mathbb{C}\langle s_1, \ldots, s_{k-1}\rangle$ and $TL_k(x) = \mathbb{C}\langle e_1, \ldots, e_{k-1}\rangle$, with additional relations

$$e_i s_i = s_i e_i = e_i, \quad e_i s_j = s_j e_i \text{ for } |i-j| > 1,$$

$$s_i e_{i+1} e_i = s_{i+1} e_i, \quad \text{and} \quad e_{i+1} e_i e_{i+1} = e_{i+1} s_i.$$

$B_k(x)$ generically centralizes $U\mathfrak{sl}_n$ in $\mathrm{End}(L(\square)^{\otimes k})$ when $x = n$.