Math 128: Lecture 25

May 22, 2014

(Very quick) introduction to quantum groups

Let q be an indeterminate. To every Lie algebra $\mathfrak{g}$ we can associate a Hopf algebra $U_q\mathfrak{g}$, called a quantum group associated to $\mathfrak{g}$, that is a *deformation* of $U\mathfrak{g}$ in the sense that $\lim_{q\to 1} U_q\mathfrak{g} = U\mathfrak{g}$.

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For example, $U\mathfrak{sl}_2$ is the algebra $\mathbb{C}[x, y, h]$ with relations

$$xy - yx = h, \quad hx - xh = 2x, \quad hy - yh = -2y. \tag{1}$$

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The associated quantum group $U_q\mathfrak{sl}_2$ is the algebra $\mathbb{C}[x, y][[h]]$ with relations

$$xy - yx = \frac{q^h - q^{-h}}{q - q^{-1}}, \quad q^h x q^{-h} = q^2 x, \quad q^h y q^{-h} = q^{-2} y. \tag{2}$$

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The first relation in (2) tends toward the first relation in (1) since

$$\lim_{q \to 1} \frac{q^h - q^{-h}}{q - q^{-1}} = \lim_{q \to 1} \frac{hq^{h-1} + hq^{-h-1}}{1 + q^{-2}} = \frac{2h}{2} = h.$$

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For the second two relations in (2), take the q derivative to get

$$hq^{h-1}xq^{-h} - q^h x h q^{-h-1} = 2qx \quad \text{and} \quad hq^{h-1}yq^{-h} - q^h y h q^{-h-1} = -2q^{-3}y,$$

which tend toward the first two relations in (1) as $q \to 1$.

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Some facts:

$$a_{ii} = \langle \beta_i^\vee, \beta_i \rangle = 2 \qquad \text{for } i = 1, \ldots, r$$

$$a_{ij} = 0, -1, -2, \text{ or } -3 \qquad \text{for } i \neq j,$$

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Identifying $\mathfrak{h}$ and $\mathfrak{h}^*$, let $B^\vee = \{h_{\beta^\vee} \mid \beta \in B\}$ and $P^\vee = \mathbb{Z}B^\vee$, so that

$$P = \mathbb{Z}\Omega = \{\lambda \in \mathfrak{h}^* \mid \lambda(h) \in \mathbb{Z} \text{ for all } h \in P^\vee\}.$$

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For $n \in \mathbb{Z}_{\ge 0}$, define

$$[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}} \qquad \text{and} \qquad [n]_x! = [n]_x[n-1]_x\cdots[1]_x,$$

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$$\begin{bmatrix} m \\ n \end{bmatrix}_x = \frac{[m]_x!}{[n]_x![m-n]_x!}$$

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For $n \in \mathbb{Z}_{\geq 0}$, define

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Let $h_i = h_{\beta_i^\vee}$, $x_i = x_{\beta_i} \in \mathfrak{g}_{\beta_i}$, and $y_i = y_{\beta_i} \in \mathfrak{g}_{-\beta_i}$.

The Lie algebra $\mathfrak{g}$ is determined by the Cartan matrix (a_{ij}) together with R in the sense that $\mathfrak{g}$ is the Lie algebra generated by $\{h_i, x_i, y_i \mid i = 1, \ldots, r\}$ with relations

1. $[h, h'] = 0$ for all $h, h' \in P^\vee$;
2. $[x_i, y_j] = \delta_{ij} h_i$;
3. $[h, x_i] = \beta_i(h) x_i$ for all $h \in P^\vee$;
4. $[h, y_i] = -\beta_i(h) y_i$ for all $h \in P^\vee$;
5. $\mathrm{ad}_{x_i}^{1-a_{ij}} x_j = 0$ for $i \neq j$; and
6. $\mathrm{ad}_{y_i}^{1-a_{ij}} y_j = 0$ for $i \neq j$.

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Let D be the diagonal matrix $(d_i)_{i=1,\ldots,r}$ symmetrizing the Cartan A (i.e. DA is symmetric). The quantum group $U_q\mathfrak{g}$ is the algebra generated by $\{q^{h_i}, x_i, y_i \mid i = 1, \ldots, r\}$ with relations

1. $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for all $h, h' \in P^\vee$; $\qquad [h, h'] = 0$
2. $x_i y_i - y_i x_i = \delta_{i,j} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}}$ where $q_i = q^{d_i}$; $\qquad [x_i, y_j] = \delta_{ij} h_i$
3. $q^h x_i q^{-h} = q^{\beta_i(h)} x_i$ for all $h \in P^\vee$; $\qquad [h, x_i] = \beta_i(h) x_i$
4. $q^h y_i q^{-h} = q^{-\beta_i(h)} y_i$ for all $h \in P^\vee$; $\qquad [h, y_i] = -\beta_i(h) y_i$
5. $\sum_{\ell=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ \ell \end{bmatrix}_{q_i} x_i^{1-a_{ij}-\ell} x_j x_i^\ell = 0$ for $i \neq j$; $\qquad \mathrm{ad}_{x_i}^{1-a_{ij}} x_j = 0$
6. $\sum_{\ell=0}^{1-a_{ij}} (-1)^\ell \begin{bmatrix} 1-a_{ij} \\ \ell \end{bmatrix}_{q_i} y_i^{1-a_{ij}-\ell} y_j y_i^\ell = 0$ for $i \neq j$. $\qquad \mathrm{ad}_{y_i}^{1-a_{ij}} y_j = 0$

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Let $K_i = q_i^{h_i} = q^{d_i h_i}$. If $\lambda = \sum_i n_i \beta_i$, let $K_\lambda = \prod_i K_i^{n_i}$.

Hopf algebra structure

The group algebra $\mathbb{C}G$ is a hopf algebra with, for $g \in G$,

comultiplication $\Delta(g) = g \otimes g$,

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The enveloping algebra $U\mathfrak{g}$ is a Hopf algebra with, for $x \in \mathfrak{g}$,

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Triangular decomposition

Recall that in $U\mathfrak{g}$,

$$U^+ \text{ is the subalgebra generated by } \{x_1, \ldots, x_r\},$$
$$U^0 \text{ is the subalgebra generated by } \mathfrak{h},$$
$$U^- \text{ is the subalgebra generated by } \{y_1, \ldots, y_r\},$$

Theorem

The enveloping algebra $U\mathfrak{g}$ has the triangular decomposition

$$U\mathfrak{g} \cong U^- \otimes U^0 \otimes U^+.$$

Likewise, let

$$U_q^+ \text{ be the subalgebra generated by } \{x_1, \ldots, x_r\},$$
$$U_q^0 \text{ be the subalgebra generated by } P^\vee,$$
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Theorem

The quantum group has the triangular decomposition

$$U_q\mathfrak{g} \cong U_q^- \otimes U_q^0 \otimes U_q^+.$$

Representations

Recall: Every finite-dimensional representation V of $U\mathfrak{g}$ is a weight module. A weight module is a highest weight module if it is generated by a weight vector v_λ^+ satisfying $U^+ v_\lambda^+ = 0$. Any highest weight module is finite-dimensional if it has highest weight in P^+. The character of V is

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Quantum version: A $U_q\mathfrak{g}$-module V^q is a weight module if

$$V^q = \bigoplus_{\mu \in P} V_\mu^q \quad \text{where} \quad V_\mu^q = \{v \in V \mid q^h v = q^{\mu(h)} v \text{ for all } h \in P^\vee\}.$$

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Take the limit $q \to 1$ [HK, §3.4]

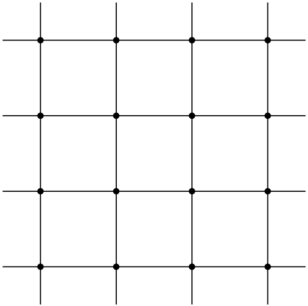
The limit as $q \to 1$ takes

$$\begin{aligned} U_q\mathfrak{g} &\to U\mathfrak{g} \\ x_i, y_i, \frac{q^h - 1}{q - 1} &\to x_i, y_i, h \\ U_q^{-,0,+} &\to U^{-,0,+} \\ L_q(\lambda) &\to L(\lambda) \quad \text{for } \lambda \in P^+ \\ \mathrm{ch}(L_q(\lambda)) &= \mathrm{ch}(L(\lambda)) \end{aligned}$$

and preserves the Hopf algebra structure.

Some physics: 6-vertex model

Consider an infinite grid



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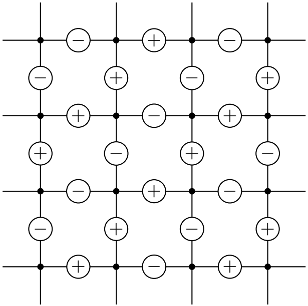

A configuration is an assignment of *spins* $\epsilon = \pm 1$ to each edge.

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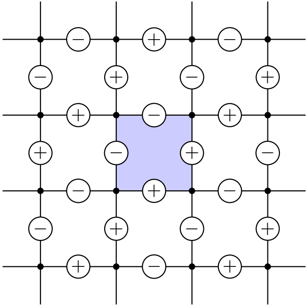
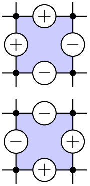


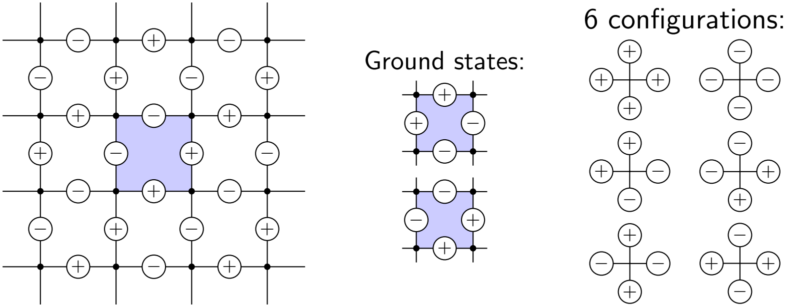
Ground states:



A configuration is an assignment of *spins* $\epsilon = \pm 1$ to each edge.For a fixed face, there are two admissible *ground state configurations*.

Some physics: 6-vertex model

Consider an infinite grid



A configuration is an assignment of *spins* $\epsilon = \pm 1$ to each edge.For a fixed face, there are two admissible *ground state configurations*. The six vertex model restricts to configurations where each vertex has one of six configurations of spins around it.

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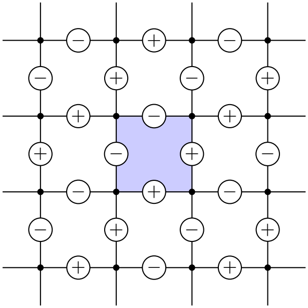

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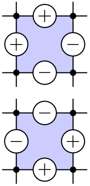
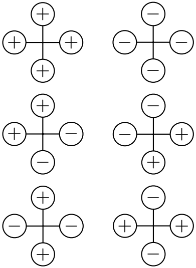


6 configurations:



A configuration is an assignment of *spins* $\epsilon = \pm 1$ to each edge.For a fixed face, there are two admissible *ground state configurations*.
The six vertex model restricts to configurations where each vertex has one of six configurations of spins around it.
Let $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$. Model a window as $V^{\otimes k}$.
Goal: Study endomorphisms of admissible configurations.

R-matrices

Quantum Yang-Baxter equation: Is there an operator R in $\mathrm{End}(V \otimes V)$ which satisfies

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} \quad \text{on } V \otimes V \otimes V,$$

where $R_{12} = R \otimes 1$ and $R_{23} = 1 \otimes R$.

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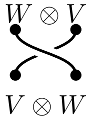
Generalized result: Existence of quantum groups, and for each quantum group, an invertible element

$$R = \sum_R R_{(1)} \otimes R_{(2)} \in U_q\mathfrak{g} \otimes U_q\mathfrak{g},$$

which yields isomorphisms

$$\check{R}_{VW} : V \otimes W \longrightarrow W \otimes V$$



that (1) satisfies braid relations, and
(2) commutes with the action on $V \otimes V$.