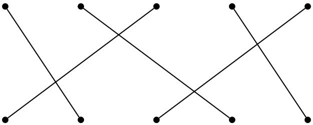
Math 128: Lecture Last

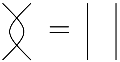
May 23, 2014

Recall: Our favorite diagram algebras so far

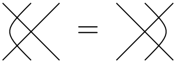
The group algebra of the symmetric group $\mathbb{C}S_k$ is the algebra with basis given by permutation diagrams



with multiplication given by concatenation, subject to the relations



and

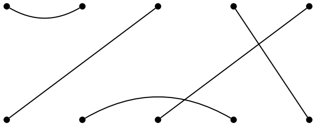


$(s_i^2 = 1)$ $\qquad$ $(s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1})$

This algebra encodes $\mathrm{End}_{U\mathfrak{sl}_n}\left(L(\square)^{\otimes k}\right)$ for $n \geq k$. (Schur - 1901)

Recall: Our favorite diagram algebras so far

The Brauer algebra $B_k(\epsilon, z)$ is the algebra with basis given by Brauer diagrams



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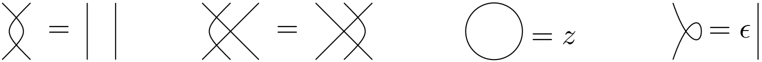
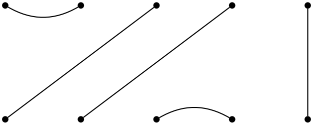


$(s_i^2 = 1)$ $(s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1})$ $(e_i^2 = z e_i)$ $(e_i s_i = \epsilon e_i,$ etc.$)$

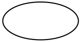
When $z = n$, this algebra encodes $\mathrm{End}_{U\mathfrak{g}}\left(L(\square)^{\otimes k}\right)$ for $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_n$, for appropriate choices of ϵ. (Brauer - 1937)

Recall: Our favorite diagram algebras so far

The Temperley-Lieb algebra $TL_k(z)$ is the algebra with basis given by non-crossing Brauer diagrams



with multiplication given by concatenation, subject to the relations

 $= z \qquad (e_i^2 = ze_i)$

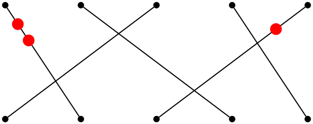
When $z = 2$, this algebra encodes $\mathrm{End}_{U\mathfrak{sl}_2}\left(L(\square)^{\otimes k}\right)$. (TL - 1971)

Recall: Our favorite diagram algebras so far

The graded Hecke algebra of type A

$$\mathbb{H}_k = \mathbb{C}[x_1, \ldots, x_k] \otimes \mathbb{C}S_k/(\text{relations})$$

is the algebra with basis given by decorated permutation diagrams with decorations north of any crossings,



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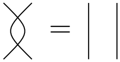
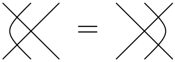
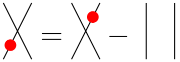




$(s_i^2 = 1)$ $\quad$ $(s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1})$ $\quad$ $(s_i x_i = x_{i+1} s_i - 1)$

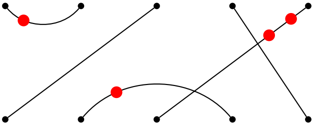
A quotient of $\mathbb{H}_k$ by certain polynomial conditions on $\mathbb{C}[\mathbf{x}]$ encodes $\mathrm{End}_{U\mathfrak{sl}_n}\left(L(\lambda) \otimes L(\square)^{\otimes k}\right)$ for $n \geq k$.
(Arakawa-Suzuki - 1998)

Recall: Our favorite diagram algebras so far

The degenerate affine Birman-Murakami-Wenzl (BMW) algebra

$$\mathbb{B}_k(\epsilon, z_0, z_1, \ldots) = \mathbb{C}[x_1, \ldots, x_k] \otimes B_k(\epsilon, z_0)/(\text{relations})$$

is the algebra with basis given by decorated Brauer diagrams with decorations north/west of any crossings or critical points,



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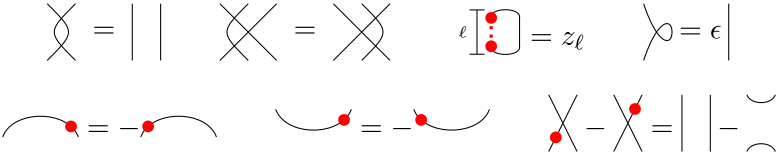


A quotient of $\mathbb{B}_k(\epsilon, \mathbf{z})$ by certain polynomial conditions on $\mathbb{C}[\mathbf{x}]$ encodes $\mathrm{End}_{U\mathfrak{g}}\left(L(\lambda) \otimes L(\square)^{\otimes k}\right)$ for $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_n$, with appropriate choices for ϵ and the z_ℓ's. (Nazarov - 1996)

Last time: Quantum groups

We saw last time one particular deformation $U_q\mathfrak{g}$ of $U\mathfrak{g}$, called the *quantum group* associated to a Lie algebra $\mathfrak{g}$. It is defined in terms of the root data of $\mathfrak{g}$, and subsequently has lots of structure similar to the structure we've seen for Lie algebras, which specializes exactly right as $q \to 1$:

$$
\begin{aligned}
q &\to 1 & & \\
U_q\mathfrak{g} &\to U\mathfrak{g} & & \text{(algebraic structure)} \\
x_i, y_i, \frac{q^h - 1}{q - 1} &\to x_i, y_i, h & & \text{(generators)} \\
U_q^{-,0,+} &\to U^{-,0,+} & & \text{(triangular decomposition)} \\
L_q(\lambda) &\to L(\lambda) \quad \text{for } \lambda \in P^+ & & \text{(highest weight modules)} \\
\mathrm{ch}(L_q(\lambda)) &= \mathrm{ch}(L(\lambda)) & & \text{(character)}
\end{aligned}
$$

This limit also preserves the Hopf algebra structure.

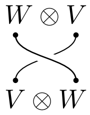
Last time: R-matrices

The existence of quantum groups came out of a study of quantum physics. Out of this study also came the existence of an invertible element

$$R = \sum_R R_{(1)} \otimes R_{(2)} \in U_q\mathfrak{g} \otimes U_q\mathfrak{g},$$

which yields isomorphisms

$$\begin{aligned} \check{R}_{VW} \colon V \otimes W &\longrightarrow W \otimes V \\ v \otimes w &\longrightarrow \sum_R R_{(2)} w \otimes R_{(1)} v \end{aligned}$$



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(1) satisfies braid relations, and
(2) commutes with the action of $U_q\mathfrak{g}$ on $V \otimes W$.

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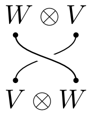

that solved the Yang-Baxter equation. Namely, it

(1) satisfies braid relations, and

(2) commutes with the action of $U_q\mathfrak{g}$ on $V \otimes W$.

Note: Recall that the coproduct structure on $U_q\mathfrak{g}$ was not symmetric.

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(x_i) = x_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(y_i) = y_i \otimes 1 + K_i \otimes y_i$$

So the action of $U_q\mathfrak{g}$ on $V^{\otimes k}$ doesn't commute with the action of S_k!

Quantum groups and braids

The quantum group produces an invertible element $R = \sum_R R_{(1)} \otimes R_{(2)} \in U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ that yields an isomorphism

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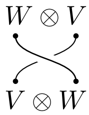

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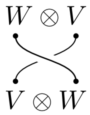
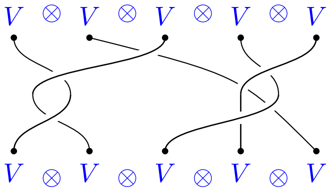


that (1) satisfies braid relations, and
(2) commutes with the $U_q\mathfrak{g}$ action on $V \otimes W$.

The braid group shares a commuting action with $U_q\mathfrak{g}$ on $V^{\otimes k}$:



Quantum groups and braids

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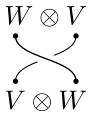
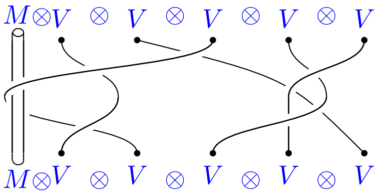


that (1) satisfies braid relations, and
(2) commutes with the $U_q\mathfrak{g}$ action on $V \otimes W$.

The one-pole/affine braid group shares a commuting action with $U_q\mathfrak{g}$ on $M \otimes V^{\otimes k}$:



Quantum groups and braids

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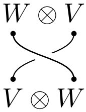
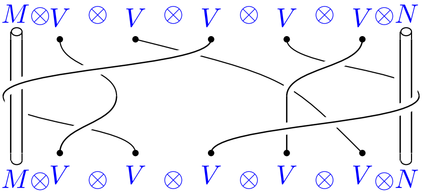


that (1) satisfies braid relations, and
(2) commutes with the $U_q\mathfrak{g}$ action on $V \otimes W$.

The two-pole braid group shares a commuting action with $U_q\mathfrak{g}$ on $M \otimes V^{\otimes k} \otimes N$:

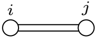


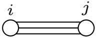
(Finite) Hecke algebras

Given a Lie algebra $\mathfrak{g}$, its Weyl group W is determined by the associated Coxeter diagram. Namely, W has generators s_i indexed by the vertices, and satisfies relations $s_i^2 = 1$ and

$s_i s_j = s_j s_i$ if 

$s_i s_j s_i = s_j s_i s_j$ if 

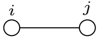
$s_i s_j s_i s_j = s_j s_i s_j s_i$ if 

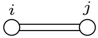
$s_i s_j s_i s_j s_i s_j = s_j s_i s_j s_i s_j s_i$ if 

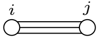
(Finite) Hecke algebras

The Hecke algebra H associated to a Weyl group W is a *deformation* of W, also determined by the associated Coxeter diagram. Namely, H has generators T_i indexed by the vertices, and satisfies relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$ for some $t_i^{1/2} \in \mathbb{C}$, and

$$T_iT_j = T_jT_i$$ if 

$$T_iT_jT_i = T_jT_iT_j$$ if 

$$T_iT_jT_iT_j = T_jT_iT_jT_i$$ if 

$$T_iT_jT_iT_jT_iT_j = T_jT_iT_jT_iT_jT_i$$ if 

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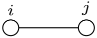
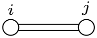
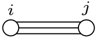
$$T_i T_j T_i = T_j T_i T_j$$ if 

$$T_i T_j T_i T_j = T_j T_i T_j T_i$$ if 

$$T_i T_j T_i T_j T_i T_j = T_j T_i T_j T_i T_j T_i$$ if 

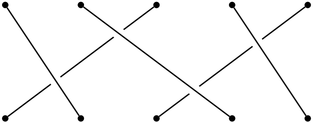
For each $w \in W$, fix a minimal length expression $w = s_{i_1} \cdots s_{i_\ell}$, and let

$$T_w = T_{i_1} \cdots T_{i_\ell}.$$

Then H has basis $\{T_w \mid w \in W\}$.

More diagram algebras

The Hecke algebra of type A_{k-1} is the algebra with basis given by permutation diagrams, each with a fixed choice of crossings



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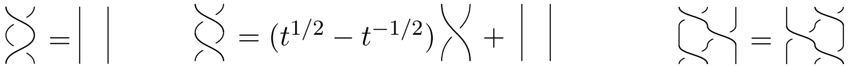
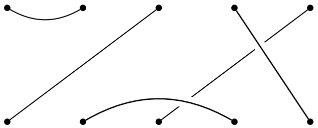


$(T_i T_i^{-1} = 1)$ $\quad ((T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0)$ $\quad (T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1})$

Then with $t_i^{1/2} = q$ generic, this algebra encodes $\mathrm{End}_{U_q\mathfrak{sl}_n}\left(L_q(\square)^{\otimes k}\right)$ for $n \geq k$. (Wenzl - 1988)

More diagram algebras

The Birman-Murakami-Wenzl (BMW) algebra $BMW_k(q,z)$ is the algebra with basis given by Brauer diagrams, each with a fixed choice of crossings



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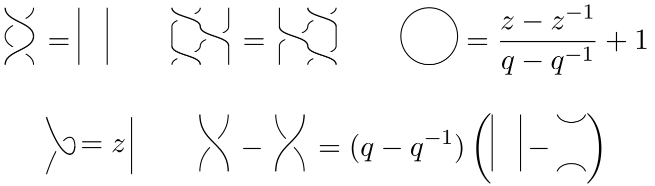

Then with q generic and good choice of z, this algebra encodes $\mathrm{End}_{U\mathfrak{g}}\left(L(\square)^{\otimes k}\right)$ for $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_n$. (BMW - 1990)

Affine Hecke algebras

Recall from our calculations of characters, we considered the algebra

$$\mathbb{C}[X] = \mathbb{C}\{X^\lambda \mid \lambda \in P\} \qquad \text{with} \quad X^\lambda X^\mu = X^{\lambda+\mu},$$

where P is the set of integral weights for $\mathfrak{g}$ given by

$$P = \mathbb{Z}\Omega \qquad \text{with} \quad \Omega = \{\omega_i \mid i = 1, \ldots, r\}.$$

This algebra is isomorphic to the Laurent polynomial ring

$$\mathbb{C}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}] \qquad \text{with} \quad X_i = X^{\omega_i - \omega_{i-1}}.$$

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$$\mathbb{C}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}] \qquad \text{with} \quad X_i = X^{\omega_i - \omega_{i-1}}.$$

The affine Hecke algebra $\mathcal{H}$ associated to a Weyl group W is

$$\mathcal{H} = \mathbb{C}[X] \otimes H$$

subject to the relations

$$T_i X^\lambda = X^{s_i\lambda} T_i + (t_i^{1/2} - t_i^{-1/2}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\beta_i}}$$

Then H has basis $\{X^\lambda T_w \mid w \in W, \lambda \in P\}$.

Affine Hecke algebra of type $\mathfrak{gl}_k$

With a little bit of work, $(T_i - t_i^{1/2})(T_i + t_i^{-1/2})$, $X_i = X^{\varepsilon_i}$, and

$$T_i X^\lambda = X^{s_i\lambda} T_i + (t_i^{1/2} - t_i^{-1/2}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\beta_i}}$$

turns into

$$T_i X_i T_i = X_{i+1} \qquad \text{and} \qquad T_1 X_1 T_1 X_1 = X_1 T_1 X_1 T_1.$$

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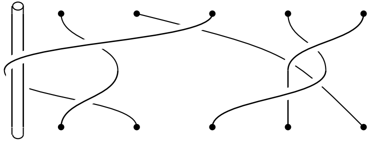
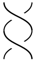
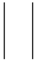
$$T_i X_i T_i = X_{i+1} \qquad \text{and} \qquad T_1 X_1 T_1 X_1 = X_1 T_1 X_1 T_1.$$

So $\mathcal{H}$ of type $\mathfrak{gl}_k$ is generated by one-pole braids



subject to relations

 $= (t^{1/2} - t^{-1/2})$  $+$  $\qquad ((T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0)$

Then with $t_i^{1/2} = q$ generic, a quotient of $\mathcal{H}$ by certain polynomial conditions on $\mathbb{C}[X]$ encodes $\mathrm{End}_{U_q\mathfrak{sl}_n}\left(L_q(\lambda) \otimes L_q(\square)^{\otimes k}\right)$ for $n \geq k$. (Orellana-Ram 2004)

Affine Hecke algebra of type $\mathfrak{gl}_k$

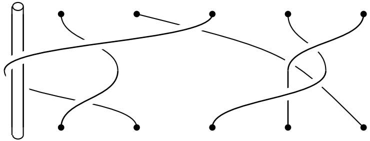
With a little bit of work, $(T_i - t_i^{1/2})(T_i + t_i^{-1/2})$, $X_i = X^{\varepsilon_i}$, and

$$T_i X^\lambda = X^{s_i\lambda} T_i + (t_i^{1/2} - t_i^{-1/2}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\beta_i}}$$

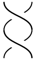
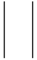
turns into

$$T_i X_i T_i = X_{i+1} \qquad \text{and} \qquad T_1 X_1 T_1 X_1 = X_1 T_1 X_1 T_1.$$

So $\mathcal{H}$ of type $\mathfrak{gl}_k$ (**and H of type B_k and C_k!!**) is generated by one-pole braids



subject to relations

 $= (t^{1/2} - t^{-1/2})$  $+$  $\quad ((T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0)$

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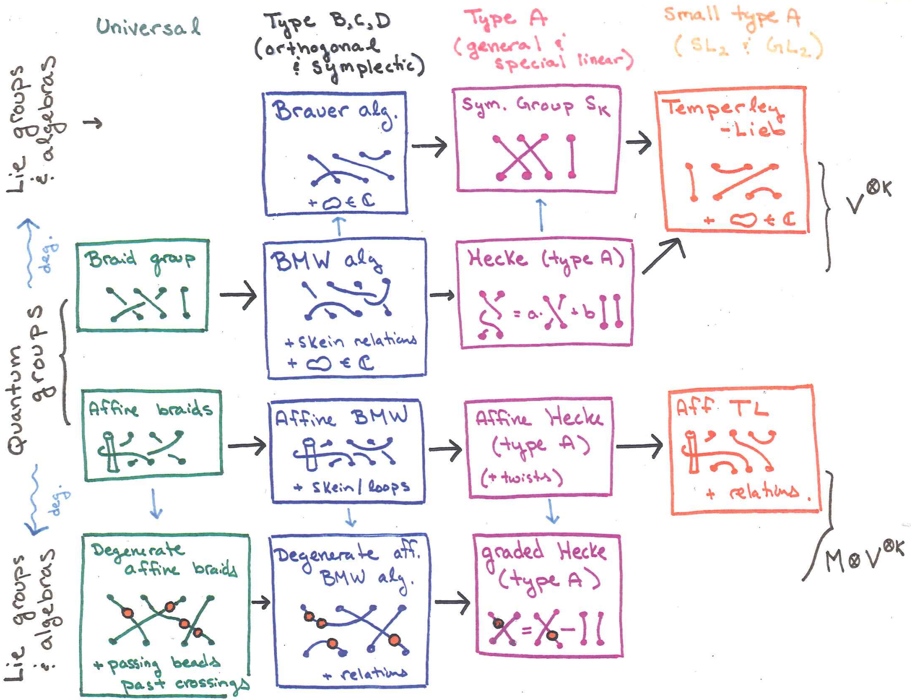
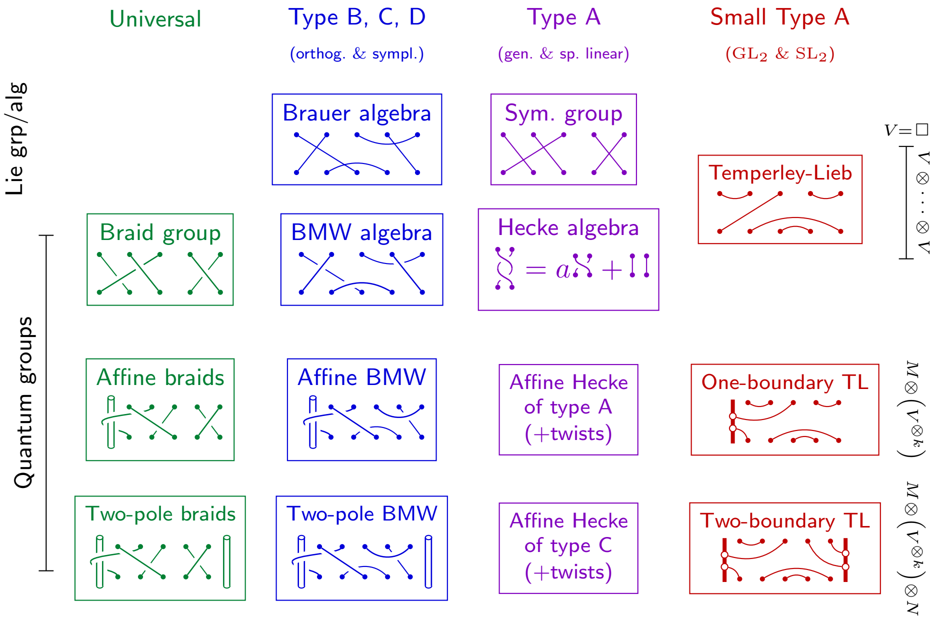


Universal
Type B,C,D (orthogonal & symplectic)
Type A (general & special linear)
Small type A (SL₂ & GL₂)
Lie groups & algebras
Brauer alg.
+ ◯ ∈ ℂ
Sym. Group S_K
Temperley-Lieb
+ ◯ ∈ ℂ
V^⊗K
deg.
Braid group
BMW alg
+ skein relations
+ ◯ ∈ ℂ
Hecke (type A)
= a · + b
Quantum groups
Affine braids
Affine BMW
+ skein / loops
Affine Hecke (type A)
(+ twists)
Aff TL
+ relations.
deg.
Degenerate affine braids
+ passing beads past crossings
Degenerate aff. BMW alg.
+ relations
graded Hecke (type A)
M⊗V^⊗K
Lie groups & algebras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Lie grp/alg
Quantum groups
Brauer algebra
Sym. group
Temperley-Lieb
$V = \square$
$V \otimes \cdots \otimes V$
Braid group
BMW algebra
Hecke algebra
$= a + $
Affine braids
Affine BMW
Affine Hecke
of type A
(+twists)
One-boundary TL
$M \otimes \left(V^{\otimes k}\right)$
Two-pole braids
Two-pole BMW
Affine Hecke
of type C
(+twists)
Two-boundary TL
$N \otimes \left(V^{\otimes k}\right) \otimes M$