Recall from last time: \( \mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C}) \) is generated by \( x, y, h \) with relations

\[
[h, x] = 2x, \quad [h, y] = -2y, \quad \text{and} \quad [x, y] = h.
\]

The universal enveloping algebra \( U\mathfrak{g} \) associated to a Lie algebra \( \mathfrak{g} \) has vector space spanned by the free group on a basis of \( \mathfrak{g} \) with relations \( ab - ba = [a, b] \) for \( a, b \in \mathfrak{g} \).

What does \( U\mathfrak{sl}_2 \) look like?
A representation of an algebra \( A \) is a vector space \( M \) (called the module) with an algebra-homomorphism \( \rho : A \to \text{End}(V) \).
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$$A \otimes M \to M,$$

where

$$ (a, m) \mapsto am = \rho(a)m$$

which is bilinear: for $c_1, c_2 \in \mathbb{C}, \ a_1, a_2 \in A, \ m_1, m_2 \in M$,

$$ (c_1 a_1 + c_2 a_2)m = c_1 a_1 m + c_2 a_2 m, \text{ and}$$

$$ a(c_1 m_1 + c_2 m_2) = c_1 am_1 + c_2 am_2$$

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Last time:

“A *representation* of a Lie algebra is a vector space $V$ together with a Lie algebra homomorphism $\rho : \mathfrak{g} \to \operatorname{End}(V)$ satisfying

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x).$$

So by definition, a representation of a Lie algebra is a representation of its enveloping algebra.
A Hopf algebra is an algebra $U$ with three maps

$$\Delta : U \rightarrow U \otimes U, \quad \varepsilon : U \rightarrow \mathbb{C}, \quad \text{and} \quad S : U \rightarrow U$$

such that

(1) If $M$ and $N$ are $U$-modules, then $M \otimes N$ with action

$$x(m \otimes n) = \sum_{x} x(1)m \otimes x(2)n$$

where $\Delta(x) = \sum_{x} x(1) \otimes x(2)$, is a $U$-module. [Note: this is called Sweedler notation]

(2) The vector space $\mathbb{C} = v\mathbb{C}$, with actions $xv_1 = \varepsilon(x)v_1$ is a $U$-module.

(3) If $M$ is a $U$-module then $M^* = \text{Hom}(M, \mathbb{C})$ with action

$$(x\varphi)(m) = \varphi(S(x)m)$$

is a $U$-module.

(4) The maps $\cup$ and $\cap$ are $U$-module homomorphisms.
Specific representations of $g$ we have so far:

1. Trivial representation: $Cv$ with $xv = 0$ for all $x \in g$.
2. Adjoint representation: $g \rightarrow \text{End}(g)$ by $x \mapsto \text{ad}_x = [\cdot, x]$.

We can get more by taking tensor products of old representations.
Specific representations of $\mathfrak{g}$ we have so far:

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(3) Standard representations of classical simple complex Lie algebras.

We can get more by taking tensor products of old representations.

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