Math 128: Lecture 3

March 28, 2014

Warmup

Recall from last time: $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ is generated by x, y, h with relations

$$[h, x] = 2x, \quad [h, y] = -2y, \quad \text{and} \quad [x, y] = h.$$

The universal enveloping algebra $U\mathfrak{g}$ associated to a Lie algebra $\mathfrak{g}$ has vector space spanned by the free group on a basis of $\mathfrak{g}$ with relations $ab - ba = [a, b]$ for $a, b \in \mathfrak{g}$.

What does $U\mathfrak{sl}_2$ look like?

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$$\begin{aligned} A \otimes M &\to M, \qquad \text{where} \\ (a, m) &\mapsto am = \rho(a)m \end{aligned}$$

which is *bilinear*: for $c_1, c_2 \in \mathbb{C}$, $a_1, a_2 \in A$, $m_1, m_2 \in M$,

$$(c_1 a_1 + c_2 a_2)m = c_1 a_1 m + c_2 a_2 m, \text{ and}$$

$$a(c_1 m_1 + c_2 m_2) = c_1 a m_1 + c_2 a m_2$$

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Last time:
"A *representation* of a Lie algebra is a vector space V together with a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{End}(V)$ satisfying $\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$."

So by definition, a representation of a Lie algebra is a representation of its enveloping algebra.

A *Hopf algebra* is an algebra U with three maps

$$\Delta : U \to U \otimes U, \qquad \varepsilon : U \to \mathbb{C}, \quad \text{and} \quad S : U \to U$$

such that

(1) If M and N are U-modules, then $M \otimes N$ with action

$$x(m \otimes n) = \sum_x x_{(1)} m \otimes x_{(2)} n$$

where $\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$, is a U-module. [Note: this is called *Sweedler notation*]

(2) The vector space $\mathbb{C} = v\mathbb{C}$, with actions $xv_1 = \varepsilon(x)v_1$ is a U-module.

(3) If M is a U-module then $M^* = \mathrm{Hom}(M, \mathbb{C})$ with action

$$(x\varphi)(m) = \varphi(S(x)m)$$

is a U-module.

(4) The maps $\cup$ and $\cap$ are U-module homomorphisms.

Specific representations of $\mathfrak{g}$ we have so far:

(1) Trivial representation: $\mathbb{C}v$ with $xv = 0$ for all $x \in \mathfrak{g}$.

(2) Adjoint representation: $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by $x \mapsto \mathrm{ad}_x = [\cdot, x]$.

(3) Standard representations of classical simple complex Lie algebras.

We can get more by taking tensor products of old representations.

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If ρ is a rep of $\mathfrak{sl}_2$, then $\rho(h)$ has at least one eigenvector v with ~~eigenvalue~~ *weight* $\lambda \in \mathbb{C}$, i.e.

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