Math 128: Lecture 5

April 2, 2014
Last time: Let $M$ be a finite-dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-module.

(1) $h$ has at least one weight vector $v \in M$. Use $hx = xh + [h, x]$ to show that $\{x^\ell v^+ \mid \ell \in \mathbb{Z}_{\geq 0}\}$ are also w.v.’s with distinct weights.

(2) Since the weights of $h$ on the $x^\ell v$’s are distinct, the non-zero $x^\ell v$’s are distinct. So since $M$ is f.d., there must be $0 \neq v^+ \in M$ with $xv^+ = 0$ and $hv^+ = \mu v^+$ for some $\mu \in \mathbb{C}$.

The vector $v^+$ is called a primitive element.

(3) Use $hy = yh + [h, y]$ to show that $\{y^\ell v^+ \mid \ell \in \mathbb{Z}_{\geq 0}\}$ are also weight vectors with distinct weights. So again, since $M$ is finite-dimensional, there must be some $d \in \mathbb{Z}_{\geq 0}$ with $y^d v^+ \neq 0$ and $y^{d+1} v^+ = 0$.

(4) Use $xy = yx + h$ to show $xy^\ell v^+ = \ell(\mu - (\ell - 1))$.

(5) Looking at the $(d+1, d+1)$ entry of $h$, use $[x, y] = h$ to show $\mu = d$.

Theorem

The simple finite dimensional $\mathfrak{sl}_2$ modules $L(d)$ are indexed by $d \in \mathbb{Z}_{\geq 0}$ with basis $\{v^+, yv^+, y^2 v^+, \ldots, y^d v^+\}$ and action $xv^+ = 0$, $y^{d+1} v^+ = 0,$

$$h(y^\ell v^+) = (d - 2\ell)(y^\ell v^+),$$

$$x(y^\ell v^+) = \ell(d + 1 - \ell)(y^{\ell-1} v^+), \quad \text{and} \quad y(y^\ell v^+) = y^{\ell+1} v^+.$$
\[ h = \begin{pmatrix} \mu & \mu - 2 & \mu - 4 & \cdots & \mu - 2d \\ \end{pmatrix} \]

\[ y = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \]

\[ x = \begin{pmatrix} 0 & \mu & 0 & 2\mu - 2 & 0 & 3\mu - 6 & \cdots & d(\mu - (d - 1)) & 0 \end{pmatrix} \]
Some facts about finite-dimensional $\mathfrak{sl}_2$ modules.

The weights of $L(d)$ are

(1) symmetric about 0,

(2) all with the same parity,

(3) are the convex hull of $\{d, -d\}$ in the lattice $2\mathbb{Z} + d$. 
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So the weights of any finite-dimensional \( \mathfrak{sl}_2 \)-module \( M \) are also symmetric about 0, with the property that

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\dim(M_{\pm a}) \leq \dim(M_{\pm b}) \quad \text{for all } 0 < b < a, \text{ with } a, b \in 2\mathbb{Z} \text{ or } 2\mathbb{Z} + 1.
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If $\{m_1, \ldots, m_r\}$ and $\{n_1, \ldots, n_s\}$ are weight bases for $\mathfrak{sl}_2$-modules $M$ and $N$ respectively, then $\{m_i \otimes n_j \mid i = 1, \ldots, r, j = 1, \ldots, s\}$ is a weight basis of $M \otimes N$, and the weight spaces of $M \otimes N$ are

$$\left(M \otimes N\right)_\gamma = \bigoplus_{\alpha + \beta = \gamma} M_\alpha \otimes M_\beta.$$
Example

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For any $d > 0$, $L(d) \otimes L(1) = L(d + 1) \oplus L(d - 1)$. So the dimension of $L(a)$ in $L(1)^{\otimes k}$ is given by the number of downward-moving paths from $L(1)$ on level on, to $L(a)$ on level $k$ in the lattice:

$k = 1$: 

$k = 2$: 

$k = 3$: 

$k = 4$: 

$k = 5$: 

...
Finite-dimensional semisimple complex Lie algebras \( g \)

Finite-dimensional: \( g \) is a finite-dimensional vector space.
Complex: \( g \) is a vector space over \( \mathbb{C} \).
Lie algebra: \( g \) is a vector space with Lie bracket \([,]\).
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- What is *semisimple*?

**Definition 1**
An *ideal* of $\mathfrak{g}$ is a subspace $\mathfrak{a}$ such that if $x \in \mathfrak{g}$, $a \in \mathfrak{a}$, then $[x, a] \in \mathfrak{a}$.
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$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_\ell$$

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An *ideal* of $\mathfrak{g}$ is a subspace $\mathfrak{a}$ such that if $x \in \mathfrak{g}$, $a \in \mathfrak{a}$, then $[x, a] \in \mathfrak{a}$. A *simple* Lie algebra is a Lie algebra with no non-trivial proper ideals and $[\mathfrak{g}, \mathfrak{g}] \neq 0$. A Lie algebra $\mathfrak{g}$ is *semisimple* if it is a direct sum of simple Lie algebras,

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as $\mathfrak{g}$-modules. A Lie algebra $\mathfrak{g}$ is *semisimple* if it has trivial center and all of the finite dimensional $\mathfrak{g}$-modules are semisimple.
Recall a *Hopf algebra* is an algebra $U$ with three maps

$$\Delta : U \rightarrow U \otimes U, \quad \varepsilon : U \rightarrow \mathbb{C}, \quad \text{and} \quad S : U \rightarrow U$$

(*coproduct, counit, and antipode*) such that

1. If $M$ and $N$ are $U$-modules, then $M \otimes N$ is a $U$-module with action

$$x(m \otimes n) = \sum_x x_1(m) \otimes x_2(n)$$

where $\Delta(x) = \sum_x x_1 \otimes x_2$.  

2. The trivial module is given by $\mathbb{C} = v\mathbb{C}$ with action

$$xv_1 = \varepsilon(x)v_1.$$  

3. If $M$ is a $U$-module then $M^* = \text{Hom}(M, \mathbb{C})$ is a $U$-module with action

$$(x\varphi)(m) = \varphi(S(x)m).$$

4. The maps $\cup : M \otimes M^* \rightarrow \mathbb{C}$ and $\cap : \mathbb{C} \rightarrow M \otimes M^*$ are $U$-module homomorphisms.
Forms

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The Killing form on a Lie algebra $g$ is $\langle x, y \rangle = \text{Tr}(\text{ad}_x \text{ad}_y)$.

If $g$ is simple, then every NIBS form is a constant multiple of the Killing form.
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