Math 128: Lecture 5

April 2, 2014

Last time: Let M be a finite-dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-module.

(1) h has at least one weight vector $v \in M$. Use $hx = xh + [h, x]$ to show that $\{x^\ell v^+ \mid \ell \in \mathbb{Z}_{\geq 0}\}$ are also w.v.'s with distinct weights.

(2) Since the weights of h on the $x^\ell v$'s are distinct, the non-zero $x^\ell v$'s are distinct. So since M is f.d., there must be $0 \neq v^+ \in M$ with

$$xv^+ = 0 \qquad \text{and} \qquad hv^+ = \mu v^+ \text{ for some } \mu \in \mathbb{C}.$$

The vector v^+ is called a *primitive element*.

(3) Use $hy = yh + [h, y]$ to show that $\{y^\ell v^+ \mid \ell \in \mathbb{Z}_{\geq 0}\}$ are also weight vectors with distinct weights. So again, since M is finite-dimensional, there must be some $d \in \mathbb{Z}_{\geq 0}$ with $y^d v^+ \neq 0$ and $y^{d+1}v^+ = 0$.

(4) Use $xy = yx + h$ to show $xy^\ell v^+ = \ell(\mu - (\ell - 1))$.

(5) Looking at the $(d+1, d+1)$ entry of h, use $[x, y] = h$ to show $\mu = d$.

Theorem

The simple finite dimensional $\mathfrak{sl}_2$ modules $L(d)$ are indexed by $d \in \mathbb{Z}_{\geq 0}$ with basis $\{v^+, yv^+, y^2v^+, \ldots, y^dv^+\}$ and action $xv^+ = 0$, $y^{d+1}v^+ = 0$,

$$h(y^\ell v^+) = (d - 2\ell)(y^\ell v^+),$$

$$x(y^\ell v^+) = \ell(d + 1 - \ell)(y^{\ell-1}v^+), \quad \textit{and} \quad y(y^\ell v^+) = y^{\ell+1}v^+.$$

$$h = \begin{pmatrix} \mu & & & & \\ & \mu - 2 & & & \\ & & \mu - 4 & & \\ & & & \ddots & \\ & & & & \mu - 2d \end{pmatrix}$$

$$y = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & \ddots & \\ & & & 1 & 0 \end{pmatrix} \quad x = \begin{pmatrix} 0 & \mu & & & \\ & 0 & 2\mu - 2 & & \\ & & 0 & 3\mu - 6 & \\ & & & \ddots & d(\mu - (d-1)) \\ & & & & 0 \end{pmatrix}$$

Some facts about finite-dimensional $\mathfrak{sl}_2$ modules.

The weights of $L(d)$ are

(1) symmetric about 0,

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So the weights of any finite-dimensional $\mathfrak{sl}_2$-module M are also symmetric about 0, with the property that

$$\dim(M_{\pm a}) \leq \dim(M_{\pm b}) \quad \text{for all } 0 < b < a \text{, with } a, b \in 2\mathbb{Z} \text{ or } 2\mathbb{Z} + 1.$$

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If $\{m_1, \ldots, m_r\}$ and $\{n_1, \ldots, n_s\}$ are weight bases for $\mathfrak{sl}_2$-modules M and N respectively, then $\{m_i \otimes n_j \mid i = 1, \ldots, r, j = 1, \ldots, s\}$ is a weight basis of $M \otimes N$, and the weight spaces of $M \otimes N$ are

$$(M \otimes N)_\gamma = \bigoplus_{\alpha+\beta=\gamma} M_\alpha \otimes M_\beta.$$

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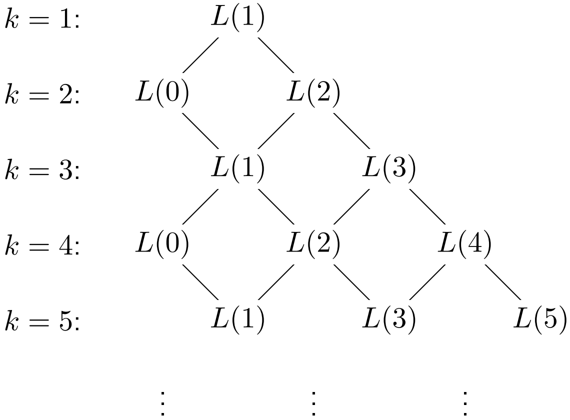
So the dimension of $L(a)$ in $L(1)^{\otimes k}$ is given by the number of downward-moving paths from $L(1)$ on level on, to $L(a)$ on level k in the lattice



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Finite-dimensional: $\mathfrak{g}$ is a finite-dimensional vector space.

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Recall a *Hopf algebra* is an algebra U with three maps

$$\Delta : U \to U \otimes U, \qquad \varepsilon : U \to \mathbb{C}, \quad \text{and} \quad S : U \to U$$

(*coproduct*, *counit*, and *antipode*) such that

(1) If M and N are U-modules, then $M \otimes N$ is a U-module with action

$$x(m \otimes n) = \sum_x x_{(1)} m \otimes x_{(2)} n$$

where $\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$.

(2) The trivial module is given by $\mathbb{C} = v\mathbb{C}$ with action $xv_1 = \varepsilon(x) v_1$.

(3) If M is a U-module then $M^* = \mathrm{Hom}(M, \mathbb{C})$ is a U-module with action

$$(x\varphi)(m) = \varphi(S(x)m).$$

(4) The maps $\cup : M \otimes M^* \to \mathbb{C}$ and $\cap : \mathbb{C} \to M \otimes M^*$ are U-module homomorphisms.

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If $\mathfrak{g}$ is simple, then every NIBS form is a constant multiple of the Killing form.