# Math 128: Lecture 8

April 9, 2014

# Some facts about roots

For $\alpha \in R$, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{ad}_h(x) = \alpha(h)x\} \neq 0$.
Let $\langle , \rangle$ be the Killing form. Recall, invariant means $\langle [x, y], z\rangle = -\langle y, [x, z]\rangle$.

1. The adjoint action of $\mathfrak{g}_\alpha$ sends $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\alpha+\beta}$.
2. If $x_\alpha \in g_\alpha$ $(\alpha \neq 0)$, then $x_\alpha$ is nilpotent.
3. If $\alpha \neq -\beta$, then $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$.
4. (Symmetry) If $\alpha \in R$, then $-\alpha \in R$.
5. The set $\{h_\alpha \mid \alpha \in R\}$ spans $\mathfrak{h}$, and so $R$ spans $\mathfrak{h}^*$.
6. If $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ then $[x_\alpha, y_\alpha] = \langle x_\alpha, y_\alpha \rangle h_\alpha$. Further, there is some $y_\alpha$ for which $\langle x_\alpha, y_\alpha \rangle \neq 0$, so $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}h_\alpha$.
7. For all $\alpha \in R$, $\langle h_\alpha, h_\alpha \rangle \neq 0$.
8. Every non-zero $x_\alpha \in \mathfrak{g}_\alpha$ is part of an $\mathfrak{sl}_2$-triple,

$$\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_{\alpha^\vee} \rangle, \quad \text{with} \quad y_\alpha \in \mathfrak{g}_{-\alpha} \text{ and } h_{\alpha^\vee} = \frac{2h_\alpha}{\langle h_\alpha, h_\alpha \rangle}.$$

# Some facts about roots

For $\alpha \in R$, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{ad}_h(x) = \alpha(h)x\} \neq 0$.
Let $\langle , \rangle$ be the Killing form. Recall, invariant means $\langle [x, y], z\rangle = -\langle y, [x, z]\rangle$.

1. The adjoint action of $\mathfrak{g}_\alpha$ sends $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\alpha+\beta}$.
2. If $x_\alpha \in g_\alpha$ ($\alpha \neq 0$), then $x_\alpha$ is nilpotent.
3. If $\alpha \neq -\beta$, then $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta\rangle = 0$.
4. (Symmetry) If $\alpha \in R$, then $-\alpha \in R$.
5. The set $\{h_\alpha \mid \alpha \in R\}$ spans $\mathfrak{h}$, and so $R$ spans $\mathfrak{h}^*$.
6. If $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ then $[x_\alpha, y_\alpha] = \langle x_\alpha, y_\alpha\rangle h_\alpha$. Further, there is some $y_\alpha$ for which $\langle x_\alpha, y_\alpha\rangle \neq 0$, so $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}h_\alpha$.
7. For all $\alpha \in R$, $\langle h_\alpha, h_\alpha\rangle \neq 0$.
8. Every non-zero $x_\alpha \in \mathfrak{g}_\alpha$ is part of an $\mathfrak{sl}_2$-triple,
$$\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_{\alpha^\vee}\rangle, \quad \text{with} \quad y_\alpha \in \mathfrak{g}_{-\alpha} \text{ and } h_{\alpha^\vee} = \frac{2h_\alpha}{\langle h_\alpha, h_\alpha\rangle}.$$
9. If $\alpha \in R$ and $c\alpha \in R$ for some $c \in \mathbb{C}^\times$, then $c = \pm 1$.

# Some facts about roots

For $\alpha \in R$, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{ad}_h(x) = \alpha(h)x\} \neq 0$.
Let $\langle , \rangle$ be the Killing form. Recall, invariant means $\langle [x, y], z\rangle = -\langle y, [x, z]\rangle$.

1. The adjoint action of $\mathfrak{g}_\alpha$ sends $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\alpha+\beta}$.
2. If $x_\alpha \in g_\alpha$ ($\alpha \neq 0$), then $x_\alpha$ is nilpotent.
3. If $\alpha \neq -\beta$, then $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta\rangle = 0$.
4. (Symmetry) If $\alpha \in R$, then $-\alpha \in R$.
5. The set $\{h_\alpha \mid \alpha \in R\}$ spans $\mathfrak{h}$, and so $R$ spans $\mathfrak{h}^*$.
6. If $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ then $[x_\alpha, y_\alpha] = \langle x_\alpha, y_\alpha\rangle h_\alpha$. Further, there is some $y_\alpha$ for which $\langle x_\alpha, y_\alpha\rangle \neq 0$, so $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}h_\alpha$.
7. For all $\alpha \in R$, $\langle h_\alpha, h_\alpha\rangle \neq 0$.
8. Every non-zero $x_\alpha \in \mathfrak{g}_\alpha$ is part of an $\mathfrak{sl}_2$-triple,
$$\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_{\alpha^\vee}\rangle, \quad \text{with} \quad y_\alpha \in \mathfrak{g}_{-\alpha} \text{ and } h_{\alpha^\vee} = \frac{2h_\alpha}{\langle h_\alpha, h_\alpha\rangle}.$$
9. If $\alpha \in R$ and $c\alpha \in R$ for some $c \in \mathbb{C}^\times$, then $c = \pm 1$.
10. For $\alpha \neq 0$, $\mathfrak{g}_\alpha = 0$ or $\mathfrak{g}_\alpha$ is one-dimensional.

# Triangular decomposition

Each semisimple finite-dimensional complex Lie algebras $\mathfrak{g}$ admit triangular decompositions:

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \quad \text{(as vector spaces)} \qquad \text{where } \mathfrak{n}^\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}_\alpha .$$

# Triangular decomposition

Each semisimple finite-dimensional complex Lie algebras $\mathfrak{g}$ admit triangular decompositions:

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \quad \text{(as vector spaces)} \qquad \text{where } \mathfrak{n}^\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}_\alpha.$$

Notice that while they are not ideals,

$$\mathfrak{n}^-, \quad \mathfrak{h}, \quad \mathfrak{n}^+, \quad \text{and} \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$$

are all subalgebras ($\mathfrak{b}$ is called the *Borel* subalgebra).

# Triangular decomposition

Each semisimple finite-dimensional complex Lie algebras $\mathfrak{g}$ admit triangular decompositions:

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \quad \text{(as vector spaces)} \qquad \text{where } \mathfrak{n}^\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}_\alpha.$$

Notice that while they are not ideals,

$$\mathfrak{n}^-, \quad \mathfrak{h}, \quad \mathfrak{n}^+, \quad \text{and} \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$$

are all subalgebras ($\mathfrak{b}$ is called the *Borel* subalgebra). On a triangular basis of $\mathfrak{g}$, for $h \in \mathfrak{h}$, $\mathrm{ad}_h$ is diagonal with entries

$$\begin{cases} 0 & \text{on basis elements in } \mathfrak{h}, \text{ and} \\ \alpha(h) & \text{on basis elements in } \mathfrak{g}_\alpha. \end{cases}$$

# Triangular decomposition

Each semisimple finite-dimensional complex Lie algebras $\mathfrak{g}$ admit triangular decompositions:

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \quad \text{(as vector spaces)} \qquad \text{where } \mathfrak{n}^\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}_\alpha.$$

Notice that while they are not ideals,

$$\mathfrak{n}^-, \quad \mathfrak{h}, \quad \mathfrak{n}^+, \quad \text{and} \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$$

are all subalgebras ($\mathfrak{b}$ is called the *Borel* subalgebra). On a triangular basis of $\mathfrak{g}$, for $h \in \mathfrak{h}$, $\mathrm{ad}_h$ is diagonal with entries

$$\begin{cases} 0 & \text{on basis elements in } \mathfrak{h}, \text{ and} \\ \alpha(h) & \text{on basis elements in } \mathfrak{g}_\alpha. \end{cases}$$

Since $\dim(\mathfrak{g}_\alpha) = 1$, for any $h_1, h_2 \in \mathfrak{h}$,

$$\langle h_1, h_2 \rangle = \sum_{\alpha \in R} \alpha(h_1)\alpha(h_2).$$

(Special to the Killing form!)

# Triangular decomposition

The triangular decomposition of $\mathfrak{g}$ induces a triangular decomposition on the enveloping algebra:

$$U\mathfrak{g} = U^- \otimes U^0 \otimes U^+ \qquad \text{with} \qquad U^\pm = U\mathfrak{n}^\pm \text{ and } U^0 = U\mathfrak{h}.$$

# Triangular decomposition

The triangular decomposition of $\mathfrak{g}$ induces a triangular decomposition on the enveloping algebra:

$$U\mathfrak{g} = U^- \otimes U^0 \otimes U^+ \qquad \text{with} \qquad U^\pm = U\mathfrak{n}^\pm \text{ and } U^0 = U\mathfrak{h}.$$

## Theorem (Birkoff-Witt)

*Let $R^+ = \{\alpha_1, \ldots, \alpha_\ell\}$ have base $B = \{\beta_1, \ldots, \beta_r\}$ . Then there are bases*

$$\left\{ y_{\alpha_1}^{m_{\alpha_1}} \cdots y_{\alpha_\ell}^{m_{\alpha_\ell}} \;\middle|\; y_\alpha \in \mathfrak{g}_{-\alpha}, m_\alpha \in \mathbb{Z}_{\geq 0} \right\} \quad \text{of } U^-,$$

$$\left\{ h_{\beta_1}^{m_{\beta_1}} \cdots h_{\beta_r}^{m_{\beta_r}} \;\middle|\; m_\beta \in \mathbb{Z}_{\geq 0} \right\} \quad \text{of } U^0, \text{ and}$$

$$\left\{ x_{\alpha_1}^{m_{\alpha_1}} \cdots x_{\alpha_\ell}^{m_{\alpha_\ell}} \;\middle|\; x_\alpha \in \mathfrak{g}_{\alpha}, m_\alpha \in \mathbb{Z}_{\geq 0} \right\} \quad \text{of } U^+.$$

*So $U\mathfrak{g}$ has basis consisting of elements*

$$y_{\alpha_1}^{m_1} \cdots y_{\alpha_\ell}^{m_\ell} h_{\beta_1}^{m'_1} \cdots h_{\beta_r}^{m'_r} x_{\alpha_1}^{m''_1} \cdots y_{\alpha_\ell}^{m''_\ell}.$$

# Some facts about roots

5. The set $\{h_\alpha \mid \alpha \in R\}$ spans $\mathfrak{h}$, and so $R$ spans $\mathfrak{h}^*$.
6. If $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ then $[x_\alpha, y_\alpha] = \langle x_\alpha, y_\alpha \rangle h_\alpha$. So $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C} h_\alpha$.
7. For all $\alpha \in R$, $\langle h_\alpha, h_\alpha \rangle \neq 0$.
8. Every non-zero $x_\alpha \in \mathfrak{g}_\alpha$ is part of an $\mathfrak{sl}_2$-triple,
$$\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_{\alpha^\vee} \rangle, \quad \text{with} \quad y_\alpha \in \mathfrak{g}_{-\alpha} \text{ and } h_{\alpha^\vee} = \frac{2h_\alpha}{\langle h_\alpha, h_\alpha \rangle}.$$
9. If $\alpha \in R$ and $c\alpha \in R$ for some $c \in \mathbb{C}^\times$, then $c = \pm 1$.
10. For $\alpha \neq 0$, $\mathfrak{g}_\alpha = 0$ or $\mathfrak{g}_\alpha$ is one-dimensional.

# Some facts about roots

5. The set $\{h_\alpha \mid \alpha \in R\}$ spans $\mathfrak{h}$, and so $R$ spans $\mathfrak{h}^*$.
6. If $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ then $[x_\alpha, y_\alpha] = \langle x_\alpha, y_\alpha \rangle h_\alpha$. So $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C} h_\alpha$.
7. For all $\alpha \in R$, $\langle h_\alpha, h_\alpha \rangle \neq 0$.
8. Every non-zero $x_\alpha \in \mathfrak{g}_\alpha$ is part of an $\mathfrak{sl}_2$-triple,
$$\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_{\alpha^\vee} \rangle, \quad \text{with} \quad y_\alpha \in \mathfrak{g}_{-\alpha} \text{ and } h_{\alpha^\vee} = \frac{2h_\alpha}{\langle h_\alpha, h_\alpha \rangle}.$$
9. If $\alpha \in R$ and $c\alpha \in R$ for some $c \in \mathbb{C}^\times$, then $c = \pm 1$.
10. For $\alpha \neq 0$, $\mathfrak{g}_\alpha = 0$ or $\mathfrak{g}_\alpha$ is one-dimensional.
11. For $\alpha, \beta \in R$,
    (a) $\beta(h_{\alpha^\vee}) \in \mathbb{Z}$,
    (b) $\beta - \beta(h_{\alpha^\vee})\alpha \in R$, and
    (c) if $\beta \neq \pm\alpha$, and $a$ and $b$ are the largest non-negative integers such that
    $$\beta - a\alpha \in R \qquad \text{and } \beta + b\alpha \in R,$$
    then $\beta + i\alpha \in R$ for all $-a \leq i \leq b$ and $\beta(h_{\alpha^\vee}) = a - b$.