1. The adjoint action of $g_\alpha$ sends $g_\beta$ to $g_{\alpha+\beta}$.
2. If $x_\alpha \in g_\alpha$ ($\alpha \neq 0$), then $x_\alpha$ is nilpotent.
3. If $\alpha \neq -\beta$, then $\langle g_\alpha, g_\beta \rangle = 0$.
4. (Symmetry) If $\alpha \in R$, then $-\alpha \in R$.
5. The set \{h_\alpha \mid \alpha \in R\} spans $\mathfrak{h}$, and so $R$ spans $\mathfrak{h}^*$.
6. If $x_\alpha \in g_\alpha$ and $y_\alpha \in g_{-\alpha}$ then $[x_\alpha, y_\alpha] = \langle x_\alpha, y_\alpha \rangle h_\alpha$.
   Further, there is some $y_\alpha$ for which $\langle x_\alpha, y_\alpha \rangle \neq 0$, so $[g_\alpha, g_{-\alpha}] = \mathbb{C}h_\alpha$.
7. For all $\alpha \in R$, $\langle h_\alpha, h_\alpha \rangle \neq 0$.
8. Every non-zero $x_\alpha \in g_\alpha$ is part of an $\mathfrak{sl}_2$-triple,
   $s_\alpha = \langle x_\alpha, y_\alpha, h_\alpha \rangle$, with $y_\alpha \in g_{-\alpha}$ and $h_\alpha = \frac{2h_\alpha}{\langle h_\alpha, h_\alpha \rangle}$.
9. If $\alpha \in R$ and $c\alpha \in R$ for some $c \in \mathbb{C}^\times$, then $c = \pm 1$.
10. For $\alpha \neq 0$, $g_\alpha = 0$ or $g_\alpha$ is one-dimensional. So if $\langle , \rangle$ is the Killing form, then for any $h_1, h_2 \in \mathfrak{h}$,
     $\langle h_1, h_2 \rangle = \sum_{\alpha \in R} \alpha(h_1)\alpha(h_2)$. 
6. If \( x_\alpha \in g_\alpha \) and \( y_\alpha \in g_{-\alpha} \) then \([x_\alpha, y_\alpha] = \langle x_\alpha, y_\alpha \rangle h_\alpha\). Further, there is some \( y_\alpha \) for which \( \langle x_\alpha, y_\alpha \rangle \neq 0 \), so \([g_\alpha, g_{-\alpha}] = \mathbb{C}h_\alpha\).

7. For all \( \alpha \in R \), \( \langle h_\alpha, h_\alpha \rangle \neq 0 \).

8. Every non-zero \( x_\alpha \in g_\alpha \) is part of an \( \mathfrak{sl}_2 \)-triple,

\[
\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_\alpha^\vee \rangle, \quad \text{with} \quad y_\alpha \in g_{-\alpha} \text{ and } h_\alpha^\vee = \frac{2h_\alpha}{\langle h_\alpha, h_\alpha \rangle}.
\]

9. If \( \alpha \in R \) and \( c\alpha \in R \) for some \( c \in \mathbb{C}^\times \), then \( c = \pm 1 \).

10. For \( \alpha \neq 0 \), \( g_\alpha = 0 \) or \( g_\alpha \) is one-dimensional. So if \( \langle , \rangle \) is the Killing form, then for any \( h_1, h_2 \in \mathfrak{h} \),

\[
\langle h_1, h_2 \rangle = \sum_{\alpha \in R} \alpha(h_1)\alpha(h_2).
\]

11. For \( \alpha, \beta \in R \),

(a) \( \beta(h_\alpha^\vee) \in \mathbb{Z} \),

(b) \( \beta - \beta(h_\alpha^\vee)\alpha \in R \), and

(c) if \( \beta \neq \pm \alpha \), and \( a \) and \( b \) are the largest non-negative integers such that

\[
\beta - a\alpha \in R \quad \text{and} \quad \beta + b\alpha \in R,
\]

then \( \beta + i\alpha \in R \) for all \(-a \leq i \leq b\) and \( \beta(h_\alpha^\vee) = a - b \).
9. If $\alpha \in R$ and $c\alpha \in R$ for some $c \in \mathbb{C}^\times$, then $c = \pm 1$.

10. For $\alpha \neq 0$, $g_\alpha = 0$ or $g_\alpha$ is one-dimensional. So if $\langle , \rangle$ is the Killing form, then for any $h_1, h_2 \in \mathfrak{h}$,

$$\langle h_1, h_2 \rangle = \sum_{\alpha \in R} \alpha(h_1)\alpha(h_2).$$

11. For $\alpha, \beta \in R$,
   (a) $\beta(h_{\alpha^\vee}) \in \mathbb{Z}$,
   (b) $\beta - \beta(h_{\alpha^\vee})\alpha \in R$, and
   (c) if $\beta \neq \pm \alpha$, and $a$ and $b$ are the largest non-negative integers such that

   $$\beta - a\alpha \in R \quad \text{and} \quad \beta + b\alpha \in R,$$

   then $\beta + i\alpha \in R$ for all $-a \leq i \leq b$ and $\beta(h_{\alpha^\vee}) = a - b$.

12. (Rationality) Let $B \subseteq R$ be a base for $R$.
   (a) $R \subseteq \mathbb{Q}B$.
   (b) For any $\alpha, \beta \in R$, $\langle \alpha, \beta \rangle \in \mathbb{Q}$.
   (c) The restriction of $\langle , \rangle$ to $\mathfrak{h}_Q^* = \mathbb{Q}B$ and $\mathfrak{h}_R^* = \mathbb{R} \otimes \mathbb{Q} \mathfrak{h}_Q^*$ is positive definite (so that $\mathfrak{h}_Q^*, \mathfrak{h}_R^*, \mathfrak{h}_Q$, and $\mathfrak{h}_R$ are all Euclidean spaces with inner product $\langle , \rangle$).
The Weyl group

Let $h_\alpha$ be the hyperplane in the real Euclidean space $h_\mathbb{R}^*$ given by

$$h_\alpha = \{ \lambda \in h_\mathbb{R}^* \mid \langle \lambda, \alpha \rangle = 0 \}.$$  

(Notice that $h_\alpha = h_{-\alpha}$.)
The Weyl group

Let $h_\alpha$ be the hyperplane in the real Euclidean space $h^*_\mathbb{R}$ given by

$$h_\alpha = \{ \lambda \in h^*_\mathbb{R} \mid \langle \lambda, \alpha \rangle = 0 \}.$$ 

(Notice that $h_\alpha = h_{-\alpha}$.)

Then $\sigma_\alpha$ extends to a map on $h^*_\mathbb{R}$, given by

$$\sigma_\alpha : h^*_\mathbb{R} \rightarrow h^*_\mathbb{R}$$

$$\lambda \mapsto \lambda - 2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \alpha,$$

which geometrically reflects weights across the hyperplane $h_\alpha$. 
The Weyl group

Let $h_\alpha$ be the hyperplane in the real Euclidean space $h^*_\mathbb{R}$ given by

$$h_\alpha = \{ \lambda \in h^*_\mathbb{R} \mid \langle \lambda, \alpha \rangle = 0 \}.$$ 

(Notice that $h_\alpha = h_{-\alpha}$.)

Then $\sigma_\alpha$ extends to a map on $h^*_\mathbb{R}$, given by

$$\sigma_\alpha : h^*_\mathbb{R} \rightarrow h^*_\mathbb{R}$$

$$\lambda \mapsto \lambda - 2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \alpha,$$

which geometrically reflects weights across the hyperplane $h_\alpha$.

The group $W$ generated by $\{ \sigma_\alpha \mid \alpha \in R^+ \}$ is called the Weyl group associated to $\mathfrak{g}$. 