Iterated Integrals and Double Integrals

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Double Integration

$S$ is described by a function $f(x, y)$ in two variables.

**Question:** What is the volume under $S$ and over $R$?
Cavalieri’s Principle – The Slicing Method

Let $B$ be a solid and $P_x$ a family of parallel planes such that
(1) $B$ lies between $P_a$ and $P_b$;
(2) the area of the slice of $B$ cut by $P_x$ is $A(x)$.
Then the volume of $B$ is equal to
$$\int_a^b A(x)\,dx.$$
Cavalieri’s Principle – The Slicing Method
Iterated Integrals

If $f$ is a continuous function and non-negative on a rectangle $R$,

$$\int \int_{R} f(x, y) dA = \int_{a}^{b} \left[ \int_{c}^{d} f(x, y) \, dy \right] \, dx$$

$$= \int_{c}^{d} \left[ \int_{a}^{b} f(x, y) \, dx \right] \, dy$$
The Double Integral

This is an operation that assigns to a function $f(x, y)$ defined and continuous over a region $D$ in the plane a number

$$\int\int_D f(x, y) \, dx\,dy$$

NOTE: If $f(x, y) \geq 0$ for all $(x, y)$ in $D$, then we can think of this number as the **volume under the graph of** $f$. 
Rectangles

The notation used for rectangles is

\[ R = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\} \]

or

\[ R = [a, b] \times [c, d] - \text{ Cartesian Product} \]
Partition of a rectangle

Suppose $R = [a, b] \times [c, d]$, a partition of $R$ is a subdivision of $R$ into smaller rectangles. You divide $[a, b]$ into $n$ equally spaced points $a = x_1 < x_2 < \ldots < x_n = b$ and $[c, d]$ into $n$ equally spaced points $c = y_1 < y_2 < \ldots , y_n = d$ and

$$x_{j+1} - x_j = \frac{b - a}{n} \quad \text{and} \quad y_{k+1} - y_k = \frac{d - c}{n}.$$
\([x_{j-1}, x_j] \times [y_{j-1}, y_j]\)

\((r, t_j)\)
Definition of Double Integral over a rectangle

The double integral is defined by

$$\int \int_R f \, dA = \lim_{\Delta x_i, \Delta y_j \to 0} \sum_{i,j=1}^{n} f(c_{ij}) \Delta x_i \Delta y_j$$

provided the limit exists. If the limit exists we say that $f$ is integrable on $R$. 
Integrability

• If $f$ is continuous on the closed interval $R$, then $\iint_{R} f(x, y)\,dA$ exists.

• If $f$ is bounded on $R$ and the set of discontinuities of $f$ has zero area, then $\iint_{R} f(x, y)\,dA$ exists.

NOTE: Here $dA = dx\,dy$ or $dA = dy\,dx$. 
Fubini’s Theorem

Let $f$ be integrable on a rectangle

$$R = [a, b] \times [c, d],$$

then $\int\int_R f(x, y) \, dA$ can be computed using the method of iterated integrals.
Properties of the Double Integral

• If $f + g$ is integrable, then
  $\iint_R (f + g) \, dA = \iint_R f \, dA + \iint_R g \, dA$;

• If $c$ is a scalar, then $\iint_R c f \, dA = c \iint_R f \, dA$;

• If $f(x, y) \leq g(x, y)$ in $R$, then
  $\iint_R f(x, y) \, dA \leq \iint_R g(x, y) \, dA$;

• If $|f|$ is integrable on $R$ then
  $|\iint_R f(x, y) \, dA| \leq \iint_R |f(x, y)| \, dA$. 