1. The graph of the function f is shown below.

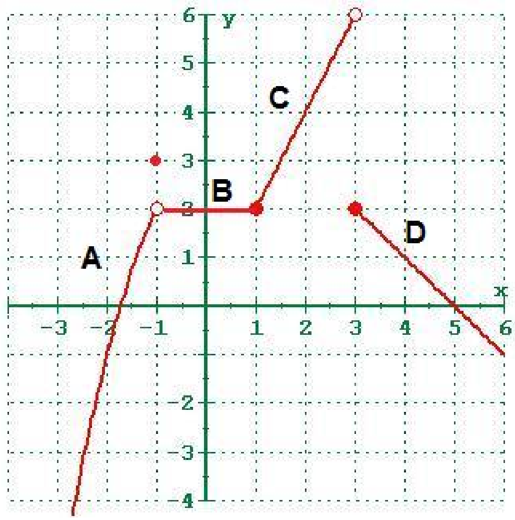


Figure 1: The graph of $f(x)$

- What is $\lim_{x\to-1^+} f(x)$? What is $\lim_{x\to-1^-} f(x)$? And does $\lim_{x\to-1} f(x)$ exist? If so, what is it, if not, why?

  Solution: As x approaches -1 from the left, we're looking at part A of the graph above. On this part of the graph, $f(x)$ goes toward the open circle at 2 as x increases toward -1, so we say that $\lim_{x\to-1^-} f(x) = 2$. Similarly, when x approaches -1 from the right, we're looking at part B of the graph above - the horizontal line segment at $y = 2$. On this part of the graph, $f(x)$ remains constantly at 2 as x decreases toward -1, so we say $\lim_{x\to-1^+} f(x) = 2$. Since the right and left limit are the same, $\lim_{x\to-1} f(x)$ does exist and $\lim_{x\to-1} f(x) = 2$.

- What is $\lim_{x\to1^+} f(x)$? What is $\lim_{x\to1^-} f(x)$? And does $\lim_{x\to1} f(x)$ exist? If so, what is it, if not, why?

  Solution: As above, when x approaches 1 from the right, we're on part C of the graph, which goes to 2 as x decreases to 1. Hence, $\lim_{x\to1^+} f(x) = 2$. When x approaches 1 from the left, we're on part B of the graph again. Now, as x increases toward 1, $f(x)$ remains constantly at 2. And so again, $\lim_{x\to1^-} f(x) = 2$. Once again, since the right and left limits agree, the limit $\lim_{x\to1} f(x)$ exists and is equal to 2.

- What is $\lim_{x\to3^+} f(x)$? What is $\lim_{x\to3^-} f(x)$? And does $\lim_{x\to3} f(x)$ exist? If so, what is it, if not, why?

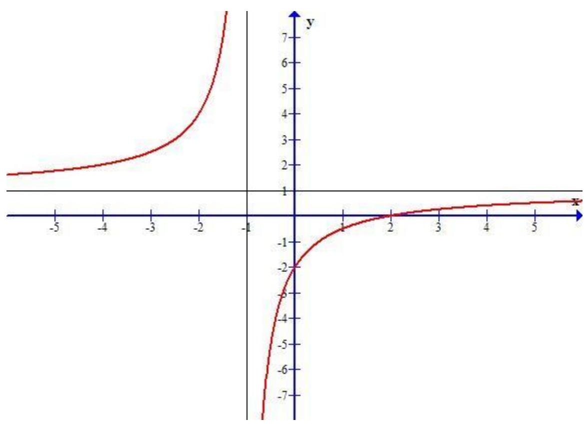
  Solution: This time, as x approaches 3 from the right, we're on part D of the graph. In this part of the graph, $f(x)$ increases to 2 as x decreases to 3, hence $\lim_{x\to3^+} f(x) = 2$.

But when x approaches 3 from the left, we're on part C of the graph again. On this part of the graph, $f(x)$ increases toward 6 as x increases toward 3, hence $\lim_{x\to 3^-} f(x) = 6$. This time the right and left limits are different and so the limit $\lim_{x\to 3} f(x)$ cannot exist.

- At which (if any) of these three points is f continuous?

  Solution: In order to be continuous at a, we must have that $\lim_{x\to a} f(x) = f(a)$ (which means $\lim_{x\to a} f(x)$ had better exist, and $f(x)$ had better be defined at a!). In the first part, $\lim_{x\to -1} f(x) = 2$, but $f(-1) = 3$, so f is not continuous at $x = -1$. In the second part $\lim_{x\to 1} f(x) = 2$ AND $f(1) = 2$, so f is continuous at $x = 1$. In the last part, $\lim_{x\to 3} f(x)$ doesn't even exist, so f can't possibly be continuous at $x = 3$.

2. Define what it means for a function to have a vertical or horizontal asymptote. Describe what this means for its graph and identify the vertical and horizontal asymptotes on the graph below.



Solution: We say $f(x)$ has a vertical asymptote at $x = a$ if $f(x)$ goes to either plus or minus infinity as x gets near a (on either side). Using limit notation: $f(x)$ has a vertical asymptote at $x = a$ if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$ (if both the right and left limit are the same, then we would have either $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$). Graphically, this means that the curve $y = f(x)$ gets very close to the line $x = a$ as x gets close to a. In the graph above, $x = -1$ serves as a vertical asymptote for the graph.

We say $f(x)$ has a horizontal asymptote at $y = L$ if $f(x)$ gets close to L as x goes to either plus or minus infinity. Using limit notation: $f(x)$ has a horizontal asymptote at $y = L$ if $\lim_{x\to\infty} f(x) = L$ or if $\lim_{x\to -\infty} f(x) = L$ (or both). Graphically, this means that the curve $y = f(x)$ gets very close to the line $y = L$ as x goes of to $\pm\infty$. In the graph above, $y = 1$ serves as a horizontal asymptote.

3. Calculate the limit $\lim_{x\to 2}(x^2 - 2x + 1/x)$. What property allows us to do this easily?

Solution: Let's call the function above f. So, $f(x) = x^2 - 2x + 1/x$. Now, since f is a rational function with $2 \in D_f$ (2 is in the domain of f), the property of Direct Substitution allows us to just plug in the value 2. So,

$$\lim_{x\to 2}(x^2 - 2x + 1/x) = f(2) = 2^2 - 2\cdot 2 + 1/2 = \frac{1}{2}$$

4. Calculate the limit $\lim_{x\to 1}\frac{x^2-1}{x-1}$. What property allows us to do this easily?

Solution: Again, let's call our function f. So, $f(x) = \frac{x^2-1}{x-1}$. Then since the numerator, x^2-1, is $x^2-1=(x-1)(x+1)$, we have

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1, \text{ for } x \neq 1$$

Let's call this function g. So, $g(x) = x+1$. Then $f(x) = g(x)$ except at $x = 1$ (where f is not defined), and so by the property on the top of page 103,

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2.$$

5. Suppose $f(x)$, $g(x)$ and $h(x)$ are defined as below. And suppose you know that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = \frac{3}{8}$. Without making any calculations and without using the nice rule we learned on October 20, what is $\lim_{x\to\infty} g(x)$? What relationship between f, g and h allows us to draw this conclusion and what theorem are we using? What line serves as a horizontal asymptote to $g(x)$?

$$\begin{aligned} f(x) &= \frac{3x^2-2x+1}{8x^2-7} \\ g(x) &= \frac{3x^2-2x+3}{8x^2-7} \\ h(x) &= \frac{3x^2-2x+8}{8x^2-7} \end{aligned}$$

Solution: Since $f(x) \leq g(x) \leq h(x)$, and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = \frac{3}{8}$, the Squeeze Theorem tells us that $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = \frac{3}{8}$. Hence, the line $y = \frac{3}{8}$ serves as a horizontal asymptote to $g(x)$.

6. Now let

$$\begin{aligned} f(x) &= \frac{x^2+6x+8}{x^2+8x+16} \\ g(x) &= \frac{x^2+5x+4}{x^2+8x+16} \\ h(x) &= \frac{x^2+7x+12}{x^2+8x+16} \end{aligned}$$

Find a relationship between these three functions that allows you to use the same theorem as in the previous problem. Use this relationship to give a relationship between $\lim_{x\to\infty} f(x)$,

$\lim_{x\to\infty} g(x)$ and $\lim_{x\to\infty} h(x)$. Evaluate each of these limits. Do these functions have any horizontal asymptotes?

Solution: This time we have $g(x) \le f(x) \le h(x)$, which means that $\lim_{x\to\infty} g(x) \le \lim_{x\to\infty} f(x) \le \lim_{x\to\infty} h(x)$. Since $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = 1$, the Squeeze Theorem says we must also have that $\lim_{x\to\infty} f(x) = 1$. And yes, these functions each have $y = 1$ as a horizontal asymptote.

7. You should know the limit laws backwards and forwards. Fill in the blanks:

$$\begin{aligned}
\lim_{x\to a}(f(x)+g(x)) &= \underline{\lim_{x\to a} f(x) + \lim_{x\to a} g(x)} \\
\lim_{x\to a} f(x) - \lim_{x\to a} g(x) &= \underline{\lim_{x\to a}(f(x)-g(x))} \\
\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} &= \underline{\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ as long as the denominator is not 0}} \\
\lim_{x\to a}(f(x)\cdot g(x)) &= \underline{\lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)} \\
c \cdot \lim_{x\to a} f(x) &= \underline{\lim_{x\to a} c \cdot f(x)} \\
\lim_{x\to a}(c) &= \underline{c} \\
\lim_{x\to a}(x) &= \underline{a} \\
\lim_{x\to a}(f(x))^r &= \underline{(\lim_{x\to a} f(x))^r}
\end{aligned}$$

8. What does it mean (in terms of the formal definitions) if $f(x)$ is continuous at $x = a$? What if I replace "continuous" with right- or left-continuous? Describe what this means for the graph of the function.

Solution: We say $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$. We say $f(x)$ is right-continuous at $x = a$ if the right limit of $f(x)$ as x approaches a is $f(a)$, that is if $\lim_{x\to a^+} f(x) = f(x)$. Similarly, $f(x)$ is left-continuous at $x = a$ if the left limit of $f(x)$ as x approaches a is $f(a)$, that is if $\lim_{x\to a^-} f(x) = f(x)$.

One way to describe continuity graphically is to say that we can draw the function near and at a without lifting our pens off the paper - this means there are no holes or jumps. Similarly, a function is right-continuous at $x = a$ if we don't need to lift our pen off the paper as we sketch the graph of $f(x)$ moving toward $x = a$ from the right (so, we're drawing the graph starting to the right of a and moving toward it) in order to reach the value $f(a)$. Likewise for the left. For example, the piecewise function in Figure 1 (from Problem 1) is right-continuous at 3 but not left-continuous.

9. You should also know how continuity is affected by those limit laws. Fill in the blank: If f and g are continuous at a and if c is some constant, then $^{(1)}\ f+g$, $^{(2)}\ f-g$, $^{(3)}\ c\cdot f$, $^{(4)}\ f\cdot g$ and $^{(5)}\ \frac{f}{g}$ are <u>continuous</u> (as long as $g(a) \neq 0$).

Choose one of (1)-(5) above and write explicitly what this means (in terms of limits). Use the limit laws to justify the equality.

(1) If $f + g$ is continuous at a, we must have $\lim_{x\to a}(f(x) + g(x)) = f(a) + g(a)$. The limit laws tell us that $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$, and since f and g are assumed to be continuous at a, $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$ and so we have that $\lim_{x\to a}(f(x) + g(x)) = f(a) + g(a)$, as desired.

The others are similar.

10. The graphs of the functions f and g are shown below. (Note: The scale on g's graph is a little odd.)

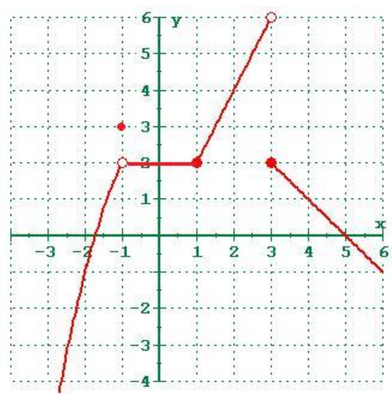


Figure 2: The graph of $f(x)$

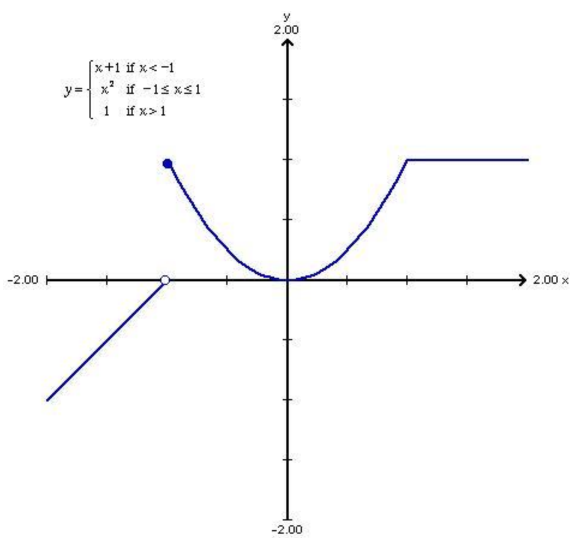


Figure 3: The graph of $g(x)$

Solution Without trying to figure out the functions involved, we know that

$$\lim_{x\to 1}(f \circ g)(x) = 2$$

Since g is continuous at 1 and f is continuous at $g(1) = 1$, because this allows us to apply Theorem 9 on page 125, which tells us that $f \circ g$ is continuous at 1 (hence, $\lim_{x\to 1}(f \circ g)(x) = (f \circ g)(1) = f(g(1)) = f(1) = 2$).

In general, the theorem states that if g is continuous at a value $x = a$ and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at $x = a$.

11. Let $f(x) = \frac{1}{x^2}$. Where does f have a horizontal asymptote? What about $g(x) = f(x+1) = \frac{1}{(x+1)^2}$ and $h(x) = f(x) + 1 = \frac{1}{x^2} + 1 = \frac{1+x^2}{x^2}$? What does this mean, in terms of limits?

Solution: We know from Theorem 5 on page 133 (and hopefully from our deductive abilities) that $f(x)$ has a horizontal asymptote at $y = 0$. Admittedly, the theorem actually states that the limit of the function as x approaches $\pm\infty$ is 0, but by our definition of a horizontal asymptote, these are really the same. Now, the graph of $g(x)$ is just the same as $f(x)$ except that it is shifted one unit to the left. But shifting the graph of $f(x)$ to the left doesn't actually chance the functions behavior at $\pm\infty$, hence $y = 0$ is also a horizontal asymptote for $g(x)$. On the other hand, the graph of $h(x)$ is the graph of $f(x)$ shifted up one unit, and this does affect the behavior of the function at $\pm\infty$ since it increases the value of $f(x)$ everywhere. So $y = 1$ is a horizontal asymptote for this function.

In terms of limits:

$$\lim_{x\to\pm\infty} g(x) = \lim_{x\to\pm\infty} f(x+1) = 0$$

and

$$\lim_{x\to\pm\infty} h(x) = \lim_{x\to\pm\infty} (f(x)+1) = \lim_{x\to\pm\infty} f(x) + \lim_{x\to\pm\infty} 1 = 0 + 1 = 1.$$

12. Justify the labeled steps using the limit laws and theorems covered in class and in your textbook.

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^2 - 2x + 7}{2x^2 - x + 3} &\overset{A}{=} \lim_{x\to\infty} \frac{\left(\frac{3x^2-2x+7}{x^2}\right)}{\left(\frac{2x^2-x+3}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{3 - 2\cdot\frac{1}{x} + 7\cdot\frac{1}{x^2}}{2 - \frac{1}{x} + 3\cdot\frac{1}{x^2}} \\
&\overset{B}{=} \frac{\lim_{x\to\infty}(3 - 2\cdot\frac{1}{x} + 7\cdot\frac{1}{x^2})}{\lim_{x\to\infty}(2 - \frac{1}{x} + 3\cdot\frac{1}{x^2})} \\
&\overset{C}{=} \frac{\lim_{x\to\infty} 3 - 2\cdot\lim_{x\to\infty}\frac{1}{x} + 7\cdot\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 2 - \lim_{x\to\infty}\frac{1}{x} + 3\cdot\lim_{x\to\infty}\frac{1}{x^2}} \\
&\overset{D}{=} \frac{3 - 2\cdot 0 + 7\cdot 0}{2 - 0 + 3\cdot 0} \\
&= \frac{3}{2}
\end{aligned}$$

Step A can be justified by the property on the top of page 103, which states that when two functions a
Step B can be justified by limit law number 5 on page 99.
Step C can be justified by limit laws 1, 2 and 3 on page 99.
Step D can be justified by Theorem 5 on page 133.

13. We found a nice rule in class for cases like the one above. We said that if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$ are polynomials (and we assume $a_n \neq 0$ and $b_m \neq 0$), then the limit $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is the same as the limit of what other (much simpler) function?

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m}$$

This means that we have the following,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \begin{cases} \frac{a_n}{b_m} \cdot \infty & \text{if } n > m \\ \frac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \end{cases}$$

Note: $\frac{a_n}{b_m} \cdot \infty$ should really just be $\pm\infty$, but the way we decide whether it should be $+$ or $-$ is by looking at the sign (+/-) of the quotient $\frac{a_n}{b_m}$. For example if $a_n = 2$ and $b_m = 3$ then their quotient is positive, and so the limit would be $+\infty$. But if $a_n = -2$ and $b_m = 3$, then their quotient is negative, and the limit would be $-\infty$.

14. Using the rule in 13, evaluate the limits:

$$\lim_{x\to\infty} \frac{x^4 - 7x + 4}{x^4 - 1} \quad \text{and} \quad \lim_{x\to\infty} \frac{-x^3 + 12x - 3}{x^2 - 2}$$

Solution:

$$\lim_{x\to\infty} \frac{x^4 - 7x + 4}{x^4 - 1} = \lim_{x\to\infty} \frac{x^4}{x^4} = 1$$

$$\lim_{x\to\infty} \frac{-x^3 + 12x - 3}{x^2 - 2} = \lim_{x\to\infty} \frac{-x^3}{x^2} = -\infty$$

15. Define the derivative of $f(x)$ at $x = a$. (Hint: There are two limits you could use here, but you should use at least one of them!) Use this definition to find $f'(2)$ when $f(x) = -3x^3 + x$ (you need to know how to take this limit, so using the shortcuts we learned on October 27 won't help you prepare for your exam). Describe what this number is in terms of the graph (Hint: you should use the word tangent).

Solution: In terms of the graph, the derivative of a function at $x = a$ is the slope of the tangent line to $y = f(x)$ at $(a, f(a))$. The first time we saw the definition of the derivative it came from an exercise which showed us that the slope of the tangent line can be seen as the limit of the slopes of secant lines originating from $(a, f(a))$. This idea gave us the following:

$$\frac{d}{dx}f(a) = f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Notice that the thing we're taking the limit of is actually just the slope of the secant line through the points $(a, f(a))$ and $(x, f(x))$, and taking this limit has the effect of moving x closer to a, which moves the secant line closer to the tangent line.

We also saw an equivalent definition of the derivative, which we liked because it allowed us to later find a general formula for the derivative of a function. This definition is:

$$\frac{d}{dx}f(a) = f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

If $f(x) = -3x^3 + x$, then $f(2) = -3\cdot 2^3 + 2 = -3\cdot 8 + 2 = -22$, so we have:

$$\begin{aligned}
f'(2) &= \lim_{h\to 0}\frac{-3(2+h)^3 + (2+h) - (-22)}{h}\\
&= \lim_{h\to 0}\frac{-3(2^3 + 3\cdot 2^2 h + 3\cdot 2\cdot h^2 + h^3) + 2 + h + 22}{h}\\
&= \lim_{h\to 0}\frac{-24 - 36h - 18h^2 - 3h^3 + h + 24}{h}\\
&= \lim_{h\to 0}\frac{-35h - 18h^2 - 3h^3}{h}\\
&= \lim_{h\to 0} -35 - 18h - 3h^2\\
&= -35
\end{aligned}$$

So the slope of the tangent line to the graph $y = -3x^3 + x$ at $(2, -22)$ is -35.

16. Define the derivative of $f(x)$ as a function of x. Use this definition to find $f'(x)$ for $f(x) = -3x^3 + x$ and verify your answer for $f'(2)$. Then find the equation for the tangent line to $f(x)$ at $(2, f(2))$.

Solution: We define the derivative of $f(x)$ as a function by taking the limit:

$$\frac{d}{dx}f(x) = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

For $f(x) = -3x^3 + x$, we have:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{-3(x+h)^3 + (x+h) - (-3x^3 + x)}{h}\\
&= \lim_{h\to 0}\frac{-3x^3 - 9x^2 h - 9xh^2 - 3h^3 + x + h + 3x^3 - x}{h}\\
&= \lim_{h\to 0}\frac{-9x^2 h - 9xh^2 - 3h^3 + h}{h}\\
&= \lim_{h\to 0} -9x^2 - 9xh - 3h^2 + 1\\
&= -9x^2 + 1
\end{aligned}$$

Using this formula, $f'(2) = -9 \cdot 2^2 + 1 = -9 \cdot 4 + 1 = -36 + 1 = -35$, which is what we got before!

17. Using the same $f(x)$ as in 15 and 16, find the values of x where $f'(x) = 0$. What does this tell us about the graph of $f(x)$? Sketch the graph of $f'(x)$. On what intervals is $f(x)$ increasing and on what intervals is $f(x)$ decreasing?

Solution: Since we fount that $f'(x) = -9x^2 + 1$, we can find the places where the derivative is 0 by setting this equal to 0 and solving for x:

$$\begin{aligned} -9x^2 + 1 &= 0 \text{ we could use the quadratic formula, but we don't need it} \\ 1 &= 9x^2 \\ \frac{1}{9} &= x^2 \\ \sqrt{\frac{1}{9}} &= x \\ \pm\frac{1}{3} &= x \end{aligned}$$

So, the points $(-\frac{1}{3}, -\frac{2}{9})$ and $(\frac{1}{3}, \frac{2}{9})$ have horizontal tangent lines.

18. What does it mean for a function g to be differentiable at a point $x = a$? How can a function fail to be differentiable? Give an example of a function which is not differentiable at $x = 1$. (A graph will suffice for this last part.)

Solution: We define the derivative of g at $x = a$ as a limit (see problem 15). We say g is differentiable at $x = a$ if this limit exists. A function can fail to be differentiable by being discontinuous or by being pointy. Both functions in problem 10 are continuous at $x = 1$, but neither is differentiable since both are pointy.

19. If the function $p(t) = 1.5t^2$ gives my position (distance in meters away from a fixed point - like a stop light) at time t (measured in seconds), find my velocity $v(t)$ and my acceleration $a(t)$. How fast am I going after 10 seconds? At $t = 10$, am I speeding up or slowing down? How can you tell?

Solution: Velocity is the derivative of positing/displacement, so $v(t) = p'(t) = 2 \cdot 1.5t = 3t$ (m/s). Acceleration is the derivative of velocity, i.e. the second derivative of position/displacement, so $a(t) = v'(t) = p''(t) = 3$ (m/s^2). My velocity after 10 seconds is given by $v(10)$, so I must be going $30m/s$ at time $t = 10$ sec. The question of whether I am speeding up or slowing down is really the question: is my velocity increasing or decreasing. When we want to know whether a function is increasing or decreasing, we look at it's derivative - in this case, that's my acceleration - if the derivative of a function is strictly positive, then the function is increasing; if it's strictly negative, then it's decreasing. Since my acceleration is constant, $a(10) = 2$, which is greater than 0, I am still speeding up.

20. Compute the following derivatives (use the shortcuts!).

(a) $f(x) = 4x^3 - 7x + 2 - x^2$. So, $f'(x) = 12x^2 - 7 - 2x$.

(b) $g(x) = 2 \cdot e^x - \frac{1}{x^2} = 2 \cdot e^x - x^{-2}$. So, $g'(x) = 2 \cdot e^x - (-2x^{-3}) = 2 \cdot e^x + \frac{2}{x^3}$.

(c) $h(x) = 3^x + \frac{3}{x}$. So, $h'(x) = \ln(3) \cdot 3^x - \frac{3}{x^2}$.

21. The Product Rule tells us that:

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \left(\frac{d}{dx} f(x)\right) \cdot g(x) + f(x) \cdot \frac{d}{dx} g(x)$$

This is the same as:

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

22. The Quotient Rule tells us that:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left(\frac{d}{dx} f(x)\right) \cdot g(x) - f(x) \cdot \frac{d}{dx} g(x)}{(g(x))^2}$$

This is the same as:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

23. Use the Product and Quotient Rules to compute the following derivatives:

(a)

$$\begin{aligned}
\frac{d}{dx}\left(e^x(4x^3 - 7x + 2 - x^2)\right) &= \frac{d}{dx} e^x \cdot (4x^3 - 7x + 2 - x^2) + e^x \cdot \frac{d}{dx}(4x^3 - 7x + 2 - x^2) \\
&= e^x \cdot (4x^3 - 7x + 2 - x^2) + e^x \cdot (12x^2 - 7 - 2x) \\
&= e^x \cdot (4x^3 - x^2 - 7x + 2 + 12x^2 - 2x - 7) \\
&= e^x \cdot (4x^3 + 11x^2 - 9x - 7)
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{d}{dx}\left(\frac{2 \cdot e^x - \frac{1}{x^2}}{x+1}\right) &= \frac{d}{dx}\left(\frac{2 \cdot e^x - x^{-2}}{x+1}\right) \\
&= \frac{\frac{d}{dx}\left(2 \cdot e^x - x^{-2}\right) \cdot (x+1) - (2 \cdot e^x - x^{-2}) \cdot \frac{d}{dx}(x+1)}{(x+1)^2} \\
&= \frac{(2 \cdot e^x - (-2)x^{-3}) \cdot (x+1) - (2 \cdot e^x - x^{-2}) \cdot 1}{(x+1)^2} \\
&= \frac{(2 \cdot e^x + 2 \cdot x^{-3}) \cdot (x+1) - (2 \cdot e^x - x^{-2}) \cdot 1}{(x+1)^2} \\
&= \frac{2x \cdot e^x + 2x \cdot x^{-3} + 2 \cdot e^x + 2 \cdot x^{-3} - 2 \cdot e^x + x^{-2}}{(x+1)^2} \\
&= \frac{2x \cdot e^x + 3 \cdot x^{-2} + 2 \cdot x^{-3}}{(x+1)^2}
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{d}{dx}\left(\frac{3^x+\frac{3}{x}}{3^x}\right) &= \frac{d}{dx}\left(\frac{3^x+3\cdot x^{-1}}{3^x}\right)\\
&= \frac{\frac{d}{dx}\left(3^x+3\cdot x^{-1}\right)\cdot(3^x)-\left(3^x+3\cdot x^{-1}\right)\cdot\frac{d}{dx}3^x}{(3^x)^2}\\
&= \frac{\left(\ln(3)\cdot 3^x+3\cdot(-1)\cdot x^{-2}\right)\cdot(3^x)-\left(3^x+3\cdot x^{-1}\right)\left(\ln(3)\cdot 3^x\right)}{3^{2x}}\\
&= \frac{\left(\ln(3)\cdot 3^{2x}-3\cdot x^{-2}\cdot 3^x\right)-\left(\ln(3)\cdot 3^{2x}+3\cdot x^{-1}\ln(3)\cdot 3^x\right)}{3^{2x}}\\
&= \frac{\ln(3)\cdot 3^{2x}-3\cdot x^{-2}\cdot 3^x-\ln(3)\cdot 3^{2x}-3\cdot x^{-1}\ln(3)\cdot 3^x}{3^{2x}}\\
&= \frac{-3\cdot x^{-2}\cdot 3^x-3\cdot x^{-1}\ln(3)\cdot 3^x}{3^{2x}}\\
&= \frac{(-3\cdot 3^x)(x^{-2}+\ln(3)\cdot x^{-1})}{3^{2x}}\\
&= \frac{(-3)(x^{-2}+\ln(3)\cdot x^{-1})}{3^x}\\
&= \frac{-1-\ln(3)\cdot x}{x^2\cdot 3^{x-1}}
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{d}{dx}\left(\left(3^x+\frac{3}{x}\right)(2x-1)\right) &= \frac{d}{dx}\left(\left(3^x+3\cdot x^{-1}\right)(2x-1)\right)\\
&= \frac{d}{dx}\left(3^x+3\cdot x^{-1}\right)\cdot(2x-1)+\left(3^x+3\cdot x^{-1}\right)\cdot\frac{d}{dx}(2x-1)\\
&= \left(\ln(3)\cdot 3^x+3\cdot(-1)\cdot x^{-2}\right)\cdot(2x-1)+\left(3^x+3\cdot x^{-1}\right)\cdot 2\\
&= \left(\ln(3)\cdot 3^x-3\cdot x^{-2}\right)\cdot(2x-1)+\left(2\cdot 3^x+6\cdot x^{-1}\right)\\
&= \left(\ln(3)\cdot 2x\cdot 3^x-6x\cdot x^{-2}\right)-\left(\ln(3)\cdot 3^x-3\cdot x^{-2}\right)+\left(2\cdot 3^x+6\cdot x^{-1}\right)\\
&= \ln(3)\cdot 2x\cdot 3^x-6\cdot x^{-1}-\ln(3)\cdot 3^x+3\cdot x^{-2}+2\cdot 3^x+6\cdot x^{-1}\\
&= \ln(3)\cdot 2x\cdot 3^x-\ln(3)\cdot 3^x+3\cdot x^{-2}+2\cdot 3^x
\end{aligned}$$

24. Given that $\frac{d}{dx}\sin(x)=\cos(x)$ and $\frac{d}{dx}\cos(x)=-\sin(x)$, use the product and/or quotient rules to find:

(a)

$$\begin{aligned}
\frac{d}{dx}\tan(x) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \\
&= \frac{\frac{d}{dx}\sin(x)\cdot\cos(x) - \sin(x)\cdot\frac{d}{dx}\cos(x)}{(\cos(x))^2} \\
&= \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{(\cos(x))^2} \\
&= \frac{(\cos(x))^2 + (\sin(x))^2}{(\cos(x))^2} \\
&= \frac{1}{(\cos(x))^2} \\
&= (\sec(x))^2
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{d}{dx}\cot(x) &= \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) \\
&= \frac{\frac{d}{dx}\cos(x)\cdot\sin(x) - \cos(x)\cdot\frac{d}{dx}\sin(x)}{(\sin(x))^2} \\
&= \frac{-\sin(x)\cdot\sin(x) - \cos(x)\cdot\cos(x)}{(\sin(x))^2} \\
&= \frac{-\left((\sin(x))^2 + (\cos(x))^2\right)}{(\sin(x))^2} \\
&= \frac{-1}{(\sin(x))^2} \\
&= -(\csc(x))^2
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{d}{dx}\sec(x) &= \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) \\
&= \frac{\frac{d}{dx}(1)\cdot\cos(x) - (1)\cdot\frac{d}{dx}\cos(x)}{(\cos(x))^2} \\
&= \frac{0\cdot\cos(x) - (-\sin(x))}{(\cos(x))^2} \\
&= \frac{\sin(x)}{(\cos(x))^2} \\
&= \sec(x)\cdot\tan(x)
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{d}{dx}\csc(x) &= \frac{d}{dx}\left(\frac{1}{\sin(x)}\right) \\
&= \frac{\frac{d}{dx}(1)\cdot\sin(x) - (1)\cdot\frac{d}{dx}\sin(x)}{(\sin(x))^2} \\
&= \frac{0\cdot\sin(x) - \cos(x)}{(\sin(x))^2} \\
&= \frac{-\cos(x)}{(\sin(x))^2} \\
&= -\csc(x)\cdot\cot(x)
\end{aligned}$$