1. (2 points) Prove that if $A$ and $B$ are independent, so are:

(a) $A$ and $B^c$.

Since $A$ and $B$ are independent, we know from a theorem that we proved in class that $P(A \cap B) = P(A) \cdot P(B)$. There are two key facts (both of which were proven in class during the second week of the term) that will help us finish this proof:

Fact 1: $P(B) + P(B^c) = 1$
Fact 2: $P(A \cap B^c) + P(A \cap B) = P(A)$

From Fact 1, we have $P(A) \cdot P(B) = P(A) - P(A) \cdot P(B^c)$. From Fact 2, we have $P(A \cap B) = P(A) - P(A \cap B^c)$. Substituting these into the equation $P(A \cap B) = P(A) \cdot P(B)$, we have

$$P(A) - P(A \cap B^c) = P(A) - P(A) \cdot P(B^c).$$

Subtracting $P(A)$ from both sides of the equation and dividing both sides by $-1$, we obtain

$$P(A \cap B^c) = P(A) \cdot P(B^c).$$

Therefore, $A$ and $B^c$ are independent (according to the theorem that we proved in class).

(b) $A^c$ and $B^c$.

The same logic shows that $A^c$ and $B^c$ are independent. Namely, we showed in part (a) that $P(A \cap B^c) = P(A) \cdot P(B^c)$. We have analogous key facts for $P(A)$ and $P(A^c)$:

Fact 1: $P(A) + P(A^c) = 1$
Fact 2: $P(A \cap B^c) + P(A^c \cap B^c) = P(B^c)$

Therefore, we have $P(A) \cdot P(B^c) = [1 - P(A^c)]P(B^c) = P(B^c) - P(A^c) \cdot P(B^c)$ from Fact 1 and $P(A \cap B^c) = P(B^c) - P(A^c \cap B^c)$. Substituting into the equation $P(A \cap B^c) = P(A) \cdot P(B^c)$, we obtain

$$P(B^c) - P(A^c \cap B^c) = P(B^c) - P(A^c) \cdot P(B^c).$$

Subtracting $P(B^c)$ from both sides and dividing by $-1$, we obtain

$$P(A^c \cap B^c) = P(A^c) \cdot P(B^c).$$

Therefore, $A^c$ and $B^c$ are independent.
2. (3 points) Give an example of 3 events which are pairwise independent but not collection-wise independent and show that the events that you choose do, indeed, have these properties.

Answers will vary. Here is one possible example:

Consider an experiment in which a fair coin is flipped twice.

Let $A =$ Event that two coin flips have the same outcome (i.e. HH or TT) 
Let $B =$ Event that the first flip lands heads 
Let $C =$ Event that the second flip lands heads 

Then $P(A) = \frac{1}{2}, P(B) = \frac{1}{2}, P(C) = \frac{1}{2}$. Moreover, we have:

$P(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(B)$.  
$P(B \cap C) = \frac{1}{4} = P(B) \cdot P(C)$.  
$P(A \cap C) = \frac{1}{4} = P(A) \cdot P(C)$.  
(In each case, HH is the only favorable outcome)

So, the events are pairwise independent according to a theorem that we proved in class. However, $P(A \cap B \cap C) = \frac{1}{4}$ (again, HH is the only favorable outcome), while $P(A) \cdot P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. Hence, the events are not collectionwise independent.
Bonus (+2 points) Consider a knight moving around on a $4 \times 4$ chessboard. We can choose any square as a starting point for the knight. Suppose that we want the knight to stop on every square exactly once (i.e. without any repeat visits). If this were possible, it would take 15 moves since the knight starts on one square and, after fifteen moves, should have visited each of the 16 squares on the $4 \times 4$ chessboard. Prove that this is impossible, regardless of knight’s starting point.

First, some vocabulary:

A Hamiltonian path is a path which visits each vertex in a graph exactly once. So, in order for the knight to visit every square on the chessboard, he would really be following a Hamiltonian path (we can think of the chessboard squares as vertices).

Solution:

In general, if we split a Hamiltonian path in 4 places (by deleting 4 vertices), at most 5 paths will be created (if you try a simple example, you’ll see why this is intuitively true). However, if we were to delete the four center vertices in the graph of the Knight’s moves, 6 paths would be created. Therefore, the knight’s path cannot be a Hamiltonian path, so it is impossible for the knight to land on every square of the chessboard without repeating any squares.