INTRODUCTION:

Buffon, a French naturalist in the XVIII century introduced a probability question for which one would have to be looking at different scenarios. “[The] problem asks to find the probability that a needle of length l will land on a line, given a floor with equally spaced parallel lines at distance d apart.”

This problem has multiple conditions. The variables we can manipulate are the length of the needle and the distance between the lines, which are very related to each other. We then can measure the angle in which the needle has fallen as well as if it has hit the line or not, which is what we are primarily interested in.

SOLUTIONS TO CASES:

To solve this problem we design different scenarios. The first one and the most straightforward is the one when the length of the needle is equal to the distance between the lines:

Let \( L \) be the length of the needle and therefore the distance between the lines in the floor. Suppose we throw the needle at an angle \( \theta \) considered to be between 0 and \( \pi \) by symmetry (if the angle is greater than \( \pi \), such as an angle \( \pi + \theta \), the needle will be in the same position as when thrown with an angle \( \theta \) itself). We can draw a line parallel to the lines on the floor, passing through the center of the needle. Let \( d \) be the distance from the center of the needle to the

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closest line. As shown in Figure 1, if \( d \leq x \) the needle hits the line, but if this is not true the needle does not hit the line. By trigonometry we can see that \( x = \frac{L}{2} \sin \theta \). We need to find the probability of that occurring. If we plot the graph of \( d = \frac{L}{2} \sin \theta \), for the specified values of \( \theta \), we can see that for the values under the sin curve \( d \) is smaller than the function and therefore the needle hits the curve. From this we can find the probability of the needle hitting the line by taking the ratio of the area under the curve with respect to the rectangle \( \frac{L}{2} \times \pi \), which represents the total amount of possible cases. We find this to be:

\[
P\left( d \leq \frac{L}{2} \sin \theta \right) = \frac{\int_{0}^{\pi} \frac{L}{2} \sin \theta \, d\theta}{\frac{L}{2} \pi} = \frac{\left[ -\frac{L}{2} \cos \theta \right]_{0}^{\pi}}{\frac{L}{2} \pi} = \frac{-\cos \pi - (-\cos 0)}{\pi} = \frac{1 + 1}{\pi} = \frac{2}{\pi}
\]

From this relation we find the probability that the needle touches the line to be 0.63662 or 63.662%.

The second scenario is when the length of the needle is smaller than the distance between the lines:

Let \( L \) be the length of the needle and \( d \) be the distance between the lines on the floor such that \( L < d \). Same way as above we find out that the probability that the needle falls on a line is:

\[
P(\text{crossing when } L < d) = \frac{\int_{0}^{\pi} \frac{L}{2} \sin \theta \, d\theta}{d \pi} = \int_{0}^{\pi} \frac{L \sin \theta}{d} \, d\theta \frac{2\pi}{2\pi d} = \frac{L}{d} \int_{0}^{\pi} \sin \theta \, d\theta = \frac{2L}{\pi d}
\]

Notice that, when \( L = d \), \( L/d = 1 \) and therefore the \( P(\text{crossing when } L = d) = \frac{2}{\pi} \), which is what we got in the first case.

The third and last scenario is when the length of the needle is bigger than the distance between the lines:

Let \( L \) be the length of the needle and \( d \) be the distance between the lines on the floor such that \( d < L \). As
seen from Figure 4, we cannot calculate this expression by using the same method as in the previous scenarios because in some cases, the ratio turns out to be greater than one, which is impossible for a probability. Therefore, there is another way to solve this issue.

Assume \( L > 1 \): We know that in order for the needle to cross the line the condition of \( d \leq \frac{L}{2} \sin \theta \) must be fulfilled as long as \( \frac{L}{2} \sin \theta \leq \frac{1}{2} \). In other words for \( 0 \leq \theta \leq \arcsin \left( \frac{1}{L} \right) \) or \( \arcsin \left( \frac{1}{L} \right) \leq \theta \leq \pi \), and for the remaining values of \( \theta \) the condition is simply \( d \leq \frac{1}{2} \). So we write \( \arcsin \left( \frac{1}{L} \right) = \alpha \) and:

\[
P(\text{crossing when } d < L) = 2 \times \frac{2}{\pi} \left[ \int_{0}^{\alpha} \frac{L}{2} \sin \theta \ d\theta + \frac{1}{2} \left( \frac{\pi}{2} - \alpha \right) \right] = \frac{2L}{\pi} (1 - \cos \alpha) + 1 - \frac{2\alpha}{\pi}
\]

Knowing that \( \cos \alpha = \cos \left( \arcsin \left( \frac{1}{L} \right) \right) = \sqrt{1 - \frac{1}{L^2}} \), we can finalize:

\[
P(\text{crossing}) = \frac{2L}{\pi} \left( 1 - \sqrt{1 - \frac{1}{L^2}} \right) + 1 - \frac{2}{\pi} \arcsin \left( \frac{1}{L} \right)
\]

After some additional manipulations to the formulas we reached the conclusion that the probability that the needle dropped crosses a line is in general:

\[
P(\text{crossing line}) = \begin{cases} \frac{2L}{\pi d} & \text{for } L \leq d \\ \frac{2}{\pi} \left( \frac{L}{d} - \sqrt{\left( \frac{L}{d} \right)^2 - 1 + \arccsc \left( \frac{L}{d} \right)} \right) & \text{for } L > d \end{cases}
\]

APPLICATIONS:

Buffon’s needle is one of the most interesting problems in Integral Probability. It has been visualized in many scenarios such as solving the situation in concentric circles or in grids of squares.

Other than having an interesting solution, this problem is considered as one of the techniques to current important issues such as problems involving cancer cells, cell growth and differentiation, military strategies, chromosome positioning etc. One of the most relevant uses is in DNA sequencing. DNA sequencing theory attempts to show analytical foundation for the natural process of DNA sequencing and gene crossing patterns (inheritance). In this field they have to calculate the probability that they cover every genome sequence and that’s where Buffon’s needle typical solution comes handy.
BIBLIOGRAPHY:


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APPENDIX:

function prob = buffon(t,L,n,m)
% t - distance between lines in the grid
% l - length of needle
% n - number of spaces
% m - number of trials to measure probability

% a set of n spaces defined by n+1 vertical lines t units apart.
% 1's represent the inside, 0's represent the lines.
A = ones(t*n+1,t*n+1);
for i=0:n
    A(:,1+t*i)=zeros(t*n+1,1);
end
imshow(A);
hold on;

V=zeros(2,2,m); % create a 3D array that will keep track of the coordinates % of the needle and the number of needles (m)

for i=1:m
    % defined the needle where vector x represents one side of the needle and % vector y represents the other
    x = randi(1+t*n,[1,2]); % generates two random points (integers) with max % being the size of the grid
    % The second side cannot be randomly selected because we are specifying a % needle length. It has to be randomly selected in such a way that it % fulfills the length requirement.
    y = x;
    while true,
        y = randi(1+t*n,[1,2]);
        if sqrt((y(1)-x(1))^2+(y(2)-x(2))^2)==L
            break;
        end;
    end;
    line([x(1),y(1)],[x(2),y(2)]);
    hold on;
    V(:,:,i)=[x;y];
end;

k = 0; % counts how many needles crossed at least a line in the grid
for i=1:m
    for j=V(1,1,i):V(2,1,i)
        if mod(j-1,t)==0 % touches the line
            k = k+1;
            break;
        end
    end
end;
prob = k/m;

Example and result:
>> prob = buffon(10,10,30,200)

prob =

0.3800